# DOCUMENT RESUME

ED 373 991 SE 054 760

| | |
|---|---|
| AUTHOR | Curcio, Frances R.; And Others |
| TITLE | Understanding Rational Numbers and Proportions. Curriculum and Evaluation Standards for School Mathematics. Addenda Series, Grades 5-8. |
| INSTITUTION | National Council of Teachers of Mathematics, Inc., Reston, Va. |
| REPORT NO | ISBN-0-87353-325-9 |
| PUB DATE | 94 |
| NOTE | 107p. |
| AVAILABLE FROM | National Council of Teachers of Mathematics, Inc., 1906 Association Drive, Reston, VA 22091-1593 (Stock No. 460, $13). |
| PUB TYPE | Guides - Classroom Use - Teaching Guides (For Teacher) (052) |
| EDRS PRICE | MF01 Plus Postage. PC Not Available from EDRS. |
| DESCRIPTORS | *Fractions; Intermediate Grades; Junior High Schools; *Mathematical Applications; Mathematics Curriculum; *Mathematics Instruction; Middle Schools; *Number Concepts; Rational Numbers; *Ratios (Mathematics); *Transformations (Mathematics); Worksheets |
| IDENTIFIERS | *Connections (Mathematics); Mathematics Activities; NCTM Curriculum and Evaluation Standards; Patterns (Mathematics); Similarities; Similar Triangles |

ABSTRACT

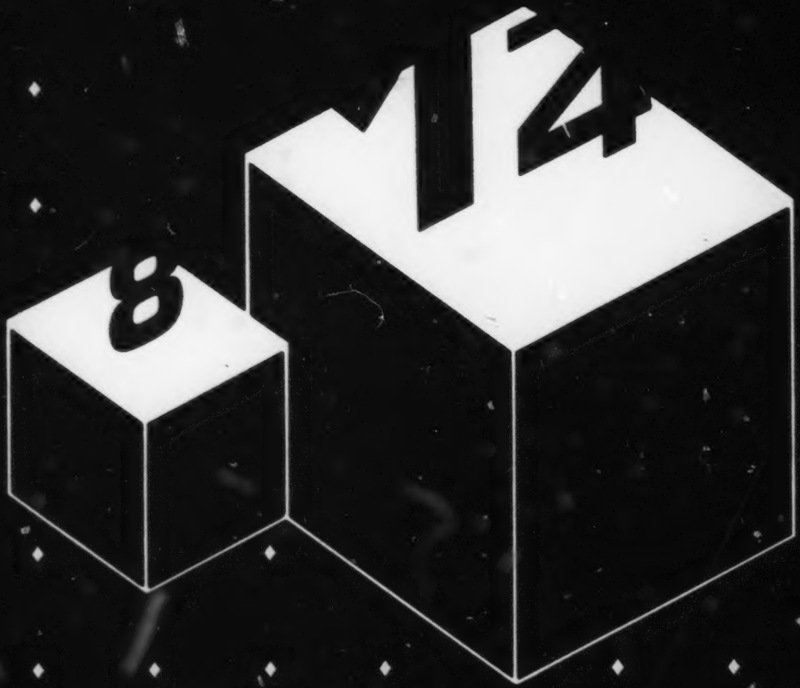
This document presents activities for use by middle school teachers which show how rational numbers and proportions are present in many real-world situations, including the collection and analysis of fractions, ratios, percents, and proportions, as well as rational number connections with geometry and similarity. The activities are divided into three content clusters: exploring and extending rational number concepts, applying rational number and proportion concepts, and making rational number connections with similarity. Each activity includes teacher notes on materials needed, teaching suggestions, lesson extensions, and ideas for integrating technology and alternative assessment. (MKR)

***********************************************************************
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
***********************************************************************

CURRICULUM AND EVALUATION

ED 373 991

# UNDERSTANDING RATIONAL NUMBERS AND PROPORTIONS



"PERMISSION TO REPRODUCE THIS MATERIAL IN OTHER THAN PAPER COPY HAS BEEN GRANTED BY
J.T. Carpenter
TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)
☒ This document has been reproduced as received from the person or organization originating it.
☐ Minor changes have been made to improve reproduction quality.
● Points of view or opinions stated in this document do not necessarily represent official OERI position or policy.

SE 054 760

NCTM

BEST COPY AVAILABLE

NCTM
MATHEMATICS:
EVERYBODY'S HERITAGE,
EVERYBODY'S FUTURE.
75th
ANNIVERSARY
NATIONAL
COUNCIL OF
TEACHERS OF
MATHEMATICS

# NEW PRODUCT ANNOUNCEMENT

For further information contact:
Tom Pearson, Advertising Assistant
(703) 620-9840 ext. 154

Reston, VA
August 1994

**Title**: *UNDERSTANDING RATIONAL NUMBERS AND PROPORTIONS: Addenda Series, Grades 5-8*

**Authors**: Frances R. Curcio, Nadine S. Bezuk, and others

**List Price**: $13. Stock #460. The book may be purchased directly from NCTM at the address below or by calling (800) 235-7566. A free catalog is available on request.

**Other Information**: Softcover, 8½" x 11", 95 pp., ISBN 0-87353-325-9. Copyright 1994.

**About the Book**: An excellent middle-school resource, brimming with classroom-ready activity sheets that show how rational numbers and proportions are present in many real-world situations. Part of the highly acclaimed Addenda to the *Curriculum and Evaluation Standards for School Mathematics* series, this book focuses on true-to-life investigations including the collection and analysis of fractions, ratios, percents and proportions, as well as rational number connections with geometry and similarity.

Each activity contains teacher notes on materials needed, teaching suggestions, lesson extensions, and ideas for integrating technology and alternative assessment.

The book is divided in three content clusters—exploring and extending rational number concepts; applying rational number and proportion concepts; and making rational number connections with similarity.

**Audience**: Mathematics teachers, supervisors, curriculum developers, and teacher educators in grades 5–8.

**For a Review Copy**: Mail the enclosed form to Tom Pearson at NCTM, Department P, 1906 Association Drive, Reston, VA 22091-1593 or fax to (703) 476-2970. For information call (703) 620-9840 ext. 154. **Please send review clippings to the address above.**

▲ ▲ ▲

CURRICULUM AND EVALUATION
STANDARDS
FOR SCHOOL MATHEMATICS
ADDENDA SERIES, GRADES 5–8

# UNDERSTANDING RATIONAL NUMBERS AND PROPORTIONS

Frances R. Curcio and Nadine S. Bezuk

*with*

| | | |
|---|---|---|
| Barbara E. Armstrong | J. Lewis McNeece | Consultants |
| Alice F. Artzt | Claire M. Newman | |
| Heidi Janzen | Francine Sicklick | Diana V. Lambdin |
| Steven T. Klass | Susan B. Turkel | Frank K. Lester, Jr. |
| Tami Martin | Judith S. Zawojewski | Kenneth P. Goldberg |

Frances R. Curcio, Series Editor

NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Copyright © 1994 by
THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, INC.
1906 Association Drive, Reston, Virginia 22091-1593
All rights reserved

Library of Congress Cataloging-in-Publication Data
Curcio, Frances R.
Understanding rational numbers and proportions / Frances R. Curcio and Nadine S. Bezuk, with Barbara E. Armstrong ... [et al.].
p. cm. — (Curriculum and evaluation standards for school mathematics addenda series. Grades 5–8)
Includes bibliographical references.
ISBN 0-87353-325-9
1. Numbers, Rational—Study and teaching (Elementary) 2. Ratio and proportion—Study and teaching (Elementary) I. Bezuk, Nadine S. II. Series.
QA135.5.C873 1994
372.7—dc20 94-15966
CIP

The publications of the National Council of Teachers of Mathematics present a variety of viewpoints. The views expressed or implied in this publication, unless otherwise noted, should not be interpreted as official positions of the Council.

Printed in the United States of America

## TABLE OF CONTENTS

**FOREWORD** .......... iv

**PREFACE** .......... v

**INTRODUCTION** .......... 1

What Are Rational Numbers? .......... 2
What Are Proportions? .......... 3
About This Book .......... 4
Moving On .......... 5

**CLUSTER A: EXPLORING AND EXTENDING RATIONAL NUMBER CONCEPTS** .......... 6

Investigation 1: Happily Handling Halves .......... 6
Investigation 2: A Trip to Benny's Bakery .......... 13

**CLUSTER B: APPLYING RATIONAL NUMBER AND PROPORTION CONCEPTS** .......... 32

Investigation 3: Garbage and Reyclying: The Rational (Number) Approach .......... 32
Investigation 4: The M&M Experiment: A Sweet Approach to Ratio and Proportion .......... 49
Investigation 5: Sharks in a Box: A Tagging-and-Recapturing Simulation .......... 57

**CLUSTER C: MAKING RATIONAL NUMBER CONNECTIONS WITH SIMILARITY** .......... 61

Investigation 6: Trips to Worlds of Shadow Creatures .......... 61
Investigation 7: What in the World Are Fractals? .......... 76

**CLOSING COMMENTS** .......... 84

**SELECTED BIBLIOGRAPHY** .......... 90

General Resources including Fractions, Ratios, and Proportions .......... 90
Similarity .......... 93
Recycling .......... 94
Fractals .......... 95

# FOREWORD

In March 1989, the National Council of Teachers of Mathematics officially released the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989). The document provides a vision and a framework for strengthening the mathematics curriculum in kindergarten to grade 12 in North American schools. Assessment, an integral part of planning for and implementing instruction, is an important feature of the document. Also presented is a contrast between traditional rote methods of teaching, which have proved to be unsuccessful, and recommendations for improving instruction supported by current educational research.

As the *Curriculum and Evaluation Standards* was being developed, it became apparent that a plethora of examples would be needed to illustrate how the vision could be realistically implemented in the K–12 classroom. A Task Force on the Addenda to the *Curriculum and Evaluation Standards for School Mathematics*, chaired by Thomas Rowan and composed of Joan Duea, Christian Hirsch, Marie Jernigan, and Richard Lodholz, was appointed by Shirley Frye, then NCTM president. The Task Force's recommendations concerning the scope and nature of the supporting publications were submitted in the fall of 1988 to the Educational Materials Committee, which subsequently framed the Addenda Project for NCTM Board approval.

Following the release of the *Curriculum and Evaluation Standards* and on the basis of recommendations of the NCTM Task Force (Rowan 1988), writing teams were assigned to develop addenda to provide teachers with classroom ideas for translating the Standards into classroom practice. Three writing teams were formed to prepare materials for grades K–6 (Miriam A. Leiva, editor), grades 5–8 (Frances R. Curcio, editor), and grades 9–12 (Christian R. Hirsch, editor).

The themes of problem solving, reasoning, communication, connections, technology, and assessment have been woven throughout the materials. The writing teams included K–12 classroom teachers, supervisors, and university mathematics educators. The materials have been field-tested in an effort to make them "teacher friendly."

Implementing the Standards will take time. The Addenda Series was written to help teachers as they undertake this task. Furthermore, the Addenda Series is appropriate for use in in-service staff development as well as for use in preservice courses in teacher education programs.

On behalf of the National Council of Teachers of Mathematics, I would like to thank all the authors, consultants, and editors who gave willingly of their time, effort, and expertise in developing these exemplary materials. In particular, I would like to acknowledge gratefully the work of Frances Curcio, Nadine Bezuk, and their contributing authors, Barbara Armstrong, Alice Artzt, Heidi Janzen, Steven Klass, Tami Martin, Lewis McNeece, Claire Newman, Francine Sicklick, Susan Turkel, and Judith Zawojewski, for preparing *Understanding Rational Numbers and Proportions*. The mathematics educators who field-tested and reviewed the investigations before publication deserve recognition. In particular, Thomas W. Allen, Agnes Cummings, Mindy Jacobson, Harry Kissileff, and Stefanie Shustek are gratefully acknowledged. Special thanks are extended to Diana Lambdin and Frank K. Lester, Jr., evaluation consultants, and Kenneth P. Goldberg, technology consultant, for their contributions to the entire Grades 5–8 Addenda Series. Finally, this project would not have materialized without the outstanding technical support supplied by Cynthia Rosso and the NCTM staff.

Bonnie H. Litwiller, *Addenda Project Coordinator*

## PREFACE

The purpose of *Understanding Rational Numbers and Proportions*, as well as of the other books in the Grades 5-8 Addenda Series, is to provide teachers with ideas and materials to support the implementation of the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989). In addition to this book, the other five publications in this series are *Patterns and Functions* (Phillips 1991), *Developing Number Sense in the Middle Grades* (Reys 1991), *Dealing with Data and Chance* (Zawojewski 1991), *Geometry in the Middle Grades* (Geddes 1992), and *Measurement in the Middle Grades* (Geddes 1994). These books are *not* an outline of a middle school curriculum, but rather, they present several topics and activities to exemplify the ideas advocated in the *Curriculum and Evaluation Standards*, and they provide examples to help students make the transition from elementary to high school mathematics.

The unifying themes of the *Curriculum and Evaluation Standards*, the characteristics of a new classroom learning environment, and the role of assessment are described below. The discussion for each topic will refer to examples in this book.

### *Unifying Themes*

The unifying themes of the *Curriculum and Evaluation Standards* include mathematics as problem solving, mathematics as communication, mathematics as reasoning, and mathematical connections. These themes are not separate, isolated entities, but rather, they are all interrelated. The ideas, examples, illustrations, and activities presented in this book are designed to demonstrate the interrelationships by weaving these themes throughout the activities as well as by providing ideas for incorporating technology and evaluation techniques.

*Mathematics as problem solving.* Although problem solving has been a goal of mathematics instruction throughout the years, it became the focus of attention with the advent of NCTM's *An Agenda for Action* (NCTM 1980). The *Curriculum and Evaluation Standards* reaffirms the importance of problem solving in mathematics instruction.

The excitement of learning and applying mathematics is generated when problems develop within the context of a situation familiar to students. Allowing them to formulate problems as they naturally arise within the context of everyday experiences gives them the opportunity to put mathematics to work, observing its usefulness and its applicability (e.g., Investigation 3). However, "not all problems require a real-world setting. Indeed, middle school students often are intrigued by story settings or those arising from mathematics itself" (NCTM 1989, p. 77). For example, see Investigation 6.

*Mathematics as communication.* Although the formal language of mathematics is concise, is high in concept density, and may be seemingly foreign, students should have the opportunity to bring meaning to mathematics on the basis of their prior knowledge and background experiences. Allowing them to talk about their experiences and how they relate to mathematics concepts, to listen to each other as they share ideas, to read mathematics in various formats (e.g., number sentences, graphs, charts), and to write about mathematical situations affords students the opportunity to compare experiences, clarify their thinking, and develop an understanding of how the mathematics they study in school is related to the mathematics they experience in the "real world." This requires the integration of the four language arts—speaking, listening, reading, and writing—with the mathematics lesson (e.g., Investigation 3).

Communicating in mathematics requires a common language and a familiarity with modes of representing mathematical ideas. Depending on students' "comfort level," oral language, prose, manipulatives, pictures, diagrams, charts, graphs, or symbols can be used to communicate ideas (e.g., Investigations 1, 2, and 3). They should be encouraged to translate from one mode to another.

*Mathematics as reasoning.* During the middle school years, students should have opportunities to develop and employ their abilities in logical and spatial reasoning (e.g., Investigations 2 and 6), as well as in proportional and graphical reasoning (e.g., Investigations 4, 5, and 6). The development of a student's reasoning ability occurs over a period of time. We can observe extreme differences between students in grade 5 and students in grade 8. As a result, instructional approaches must reflect these differences.

Depending on students' readiness, exploratory activities, experiments, and projects may require them to give a descriptive account of what they observe, an informal argument based on empirical results, or a formal proof supporting a conjecture (Hirsch and Lappan 1989). Active learners should be constantly involved in questioning, examining, conjecturing, and experimenting (e.g., Investigations 1, 2, 3, and 7).

*Mathematical connections.* Traditionally, mathematics has often been presented as an isolated set of rules to be memorized. The *Curriculum and Evaluation Standards* suggests that mathematics be presented as an integrated whole. Students should observe the interrelatedness among branches of mathematics: number theory, geometry, algebra, probability, and so on (e.g., Investigations 5, 6, and 7). Furthermore, students should become aware of how mathematics is related to such other disciplines as environmental science, biological sciences, architecture, art, literature, music, social studies, business, and industrial technology (e.g., Investigations 3 and 6).

There is no guarantee that allowing students to explore, create, and experiment within the context of a problem-solving setting will lead them to discover connections between and among mathematics concepts. Teachers may need to guide students in discovering connections and to elicit these connections explicitly. Students who recognize connections within mathematics and with other disciplines can understand and appreciate the logical unity and the power of mathematics (Steen 1989).

### A New Classroom Environment

Implementing the *Curriculum and Evaluation Standards* requires a new way of teaching. The traditional teacher roles of authority figure and information disseminator must change to learning facilitator and instructional decision maker.

Knowledge about students and how they learn mathematics can contribute to establishing a conducive learning environment for middle school students. The teacher selects the instructional objectives on the basis of knowledge of his or her students, knowledge of mathematics, and knowledge of pedagogy (NCTM 1991). After selecting the instructional objectives, the teacher must decide how to deliver the content. Is the use of manipulatives appropriate? Is the use of technology appropriate? Is a cooperative learning setting appropriate?

*Appropriate use of manipulatives.* Manipulatives are multisensory tools for learning that provide students with a means of communicating ideas by allowing them to model or represent their ideas concretely. Using manipulatives, however, does not guarantee the understanding of a mathematics concept (Baroody 1989). After allowing students to explore using manipulatives, teachers must formulate questions to elicit the important, "big" mathematical ideas that enable students to make connections between the mathematics and the manipulatives used to represent the concept(s) (e.g., Investigations 2, 5, and 7).

*Appropriate use of technology.* Developments in technology have made the traditional, computation-dominated mathematics curriculum obsolete. As a result, technology has been given a prominent place in the *Curriculum and Evaluation Standards,* in terms of which technology should be made available for use in the classroom and how technology should be used in mathematics instruction. It is expected that at the middle school level, students will have access to appropriate calculators, and computers should be available for demonstration purposes as well as for individual and group work (NCTM 1989, p. 8). It should be noted that new advances in technology are being made constantly. Teachers should keep abreast of new developments that support mathematics instruction.

In this book, suggestions are made for integrating the use of computers to organize and analyze data (e.g., Investigations 4 and 5) and to create geometric shapes (e.g., Investigation 7). Specifically, the use of Microsoft Works, Visicalc, Lotus, Pascal, and Logo are recommended in some activities.

Integrating the use of calculators is also recommended. In particular, calculators with and without fraction capability are employed to examine relationships (e.g., Investigation 3), to facilitate and expedite computation (e.g., Investigations 4 and 5), and to test conjectures (Investigations 1 and 7).

*Appropriate use of cooperative learning groups.* Traditionally, mathematics has been taught as a "solo," isolated activity, yet in business and industry, mathematicians often work in teams to solve problems and attain common objectives (Steen 1989). Allowing students to work in cooperative groups affords them the opportunity to develop social and communication skills while working with peers of various ethnic, religious, and racial groups.

Cooperative learning environments, characterized by students working together and interacting with each other, contribute to internalizing concepts by forcing the students to defend their views against challenges brought by their peers. The value of this approach is supported by the work of Vygotsky ([1934]1986) who discussed the increasingly interrelated nature of language and cognition as children grow.

Cooperative groups usually contain three to five students and may be established for various lengths of time (Artzt and Newman 1990; Davidson 1989). Unlike most traditional small-group instruction in reading and mathematics, cooperative learning groups are heterogeneous, and everyone must work together for the common good of all. Students who understand the concept being discussed are responsible for explaining it to those who do not understand. Examples for using cooperative learning can be found in Investigations 3, 4, 5, and 6. When using cooperative groups, teachers must consider new ways of assessing student performance to ensure the success of the instructional objective(s).

### The Role of Assessment

Making changes in the content and methods of mathematics instruction will also require making changes in why and how students' work is assessed. Assessment should be an integral part of instruction and not be limited to grading and testing. According to Kroll and Lester (1991), there are at least four reasons for collecting assessment information:

- To make decisions about the content and methods of mathematics instruction
- To make decisions about classroom climate
- To help in communicating what is important
- To assign grades

In other words, assessment includes much more than marking right and wrong answers. It "must be more than testing; it must be a continuous, dynamic, and often informal process" (NCTM 1989, p. 203). What methods can be used for assessment purposes? Teachers may consider a variety of types of assessment: diagnostic activities, embedded activity assessments, embedded writing prompts, and projects and performance assessments. (For details and other ideas, see Stenmark [1991] and NCTM [1993].)

*Diagnostic activities* help determine the students' grasp of prerequisites for an activity or for learning a new concept. Students' responses to brief questions posed before working on an activity or their responses in reviewing old concepts and making connections may reveal gaps or weaknesses in their understanding or skill. Investigations 1 and 7 contain diagnostic activities.

*Embedded assessment* refers to questions, exercises, or problems that are embedded within the larger context of a lesson, an exploration, or an activity and that are an integral part of the activity but that are highlighted to indicate that a teacher might focus attention on them to determine students' levels of understanding or skill. Examples of embedded assessments, which generally use a holistic scoring rubric to evaluate students' written work, can be found in Investigations 2 and 6.

*Embedded writing prompts* are statements or questions posed for students to respond to in writing. If students keep mathematics journals, these statements or questions might be termed *journal prompts*. Examples of writing prompts can be found in Investigations 1 and 2.

*Projects and performance assessments* require students to apply concepts and skills to solve problems adapted from real-world situations in which mathematics comes into play. Assessing the quality of this work is often complex. Teachers may use an analytic-holistic scoring rubric that would involve a holistic rubric for three or more aspects of the solution effort (e.g., initial understanding, development of a plan, execution of the plan, and quality of the final solution). The five-part project in Investigation 3 is an example of a project assessment. Examples of performance assessment tasks can be found in Investigations 3 and 7.

This brief description of the unifying themes of the *Curriculum and Evaluation Standards*, the characteristics of a new learning environment, and the role of assessment is a starting point to understand and to appreciate the ideas that are presented in this book. It is hoped that these ideas build a foundation for developing rational number and proportion concepts in the middle grades and generate interest among teachers for improving instructional techniques in mathematics.

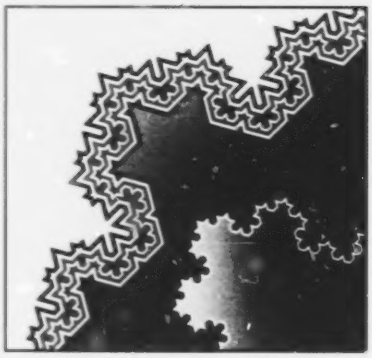
Frances R. Curcio, Editor
Grades 5–8 Addenda Series

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

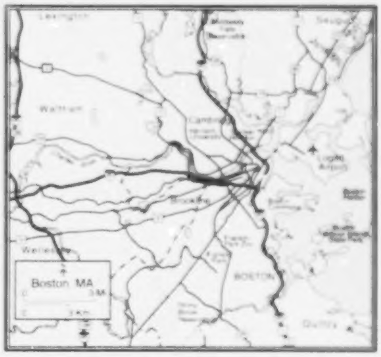
## INTRODUCTION

Personal encounters with real-world number situations help us recognize the importance of rational numbers. Although the whole number system is useful and important as a tool for counting and calculating, we eventually realize that "different" numbers are needed for such measurements as a part of a liter of milk, a part of a dollar, a part of a meter of ribbon, one of two equal shares of a piece of cake, and the likelihood of rain or of winning a lottery. To describe these quantities, the whole number system is inadequate. Different types of numbers—rational numbers—are needed.

Although not limited to the situations mentioned above, rational numbers, taking the form of fractions, ratios, decimals, and percents, mathematically represent the situations described. Historically, the development of fractions provided a means to make a transition from counting to measuring. Furthermore, fractions were invented to deal with parts of things, which take the form of discrete sets (e.g., a dozen eggs) or continuous quantities (e.g., a square meter of material), and to make division always possible (e.g., sharing two apples fairly among three children: 2 ÷ 3, or 2/3). As mathematics continues to develop as a discipline, further uses of rational numbers will be found. For example, probability theory formulated by Blaise Pascal and Pierre de Fermat during the seventeenth century and fractal geometry formulated by Benoit Mandelbrot in the twentieth century employ rational numbers.



Quantitative, proportional relationships also permeate the world around us. Music, nature, probability, shopping, money, recipes, maps, and so on, provide settings replete with proportional relationships. Many of these settings are naturally interesting for middle school students. In more formal school settings, the study of algebra, geometry, and some aspects of biology, chemistry, and physics builds on concepts related to proportionality (Hoffer and Hoffer 1992, pp. 303–5).



Rational numbers and proportions are a major focus of the mathematics curriculum in the middle grades. The importance of students' developing a strong rational-number-and-proportion foundation for maximizing career options cannot be overstated. Carefully planned instruction that builds on students' natural language and informal notions of fractions, ratios, decimals, percents, and proportions will help learners become quantitatively literate citizens as they apply concepts of rational numbers and proportions in their daily living. The development of these complex understandings evolves over a period of time. Opportunities to develop this skill will give students the tools for further study in the mathematical and natural sciences.

The importance of the rational number system is highlighted in several different parts of the *Curriculum and Evaluation Standards for School Mathematics* for grades 5 to 8, including the standards on number and number relationships, number systems and number theory, and computation and estimation. In particular, the *Curriculum and Evaluation Standards* recommends that middle school students develop the facility to translate between and among equivalent numerical forms (e.g., fractions, decimals, and percents) and investigate relationships among these different forms; develop number sense for rational numbers; apply ratios, proportions, and percents in a variety of contexts; "develop, analyze, and explain methods for solving proportions"; and "use computa-

***"Rational-number concepts are among the most complex and important mathematical ideas children encounter during their presecondary school years" (Behr, Lesh, Post, and Silver 1983, p. 91).***

tion, estimation, and proportions to solve problems" (NCTM 1989, pp. 87, 94). Furthermore, solutions to many problems in measurement, geometry, algebra, probability, and statistics require knowledge of, and facility with, rational numbers and proportions. And, these ideas provide a foundation for analyzing patterns and functions. The investigations in this book present some ideas for developing rational-number-and-proportion concepts in the context of problem-solving situations that build on making connections within and beyond mathematics, as well as creating opportunities for students to reason and communicate mathematically. A discussion of rational numbers, proportions, and the activities provided in this book follows.

### What Are Rational Numbers?

A rational number, commonly thought of as a "fraction," is a number that can be expressed as a quotient or a ratio of two integers, *a* and *b*; that is, *a*/*b*, where *b* does not equal zero. The investigations in this book will focus on the nonnegative rational numbers, illustrated within the context of several different interpretations.

Among the several different ways to interpret rational numbers are a part-whole meaning, a quotient meaning, a ratio meaning, and an operator meaning (Kieren 1980, 1988). Examples from the investigations in this book highlight each of these interpretations.

***"The part-whole notion of rational numbers is fundamental to the other interpretations" (Behr and Post 1992, p. 203).***

*Part-whole meaning.* In the part-whole meaning, a unit in the form of a continuous shape (e.g., a cake) or a discrete set (e.g., a number of cookies) is introduced. The unit is partitioned into equal-sized parts or noncongruent shapes with equal area. For example, a sheet cake is to be cut into eight equal-sized pieces. What will be the size of each piece? (See Investigation 2, Activity 1.) The cake will be partitioned into eight equal pieces (either congruent pieces or pieces equal in area), each piece being 1/8 of the whole cake.

*Quotient meaning.* The quotient meaning, where a number of objects needs to be shared or divided equally into a number of groups, appears more frequently in applications than the other meanings (Lay 1982). An example follows. "You bought a baker's dozen of cookies (13 cookies) that you want to share equally with the members of your family. How many cookies will each person get?" (See Investigation 2, Activity 3.) There are thirteen objects, that is, cookies, to be shared or divided equally among a certain number of people in your family (e.g., 4 people). This situation is represented as 13 ÷ 4, or 13/4, or 3 and 1/4 cookies for each person.

*Ratio meaning.* A ratio is a relationship between two quantities of the same measure. For example, "What is the ratio of each color of M&Ms to the total number of M&Ms in a bag?" (See Investigation 4, Activity 1.) Depending on the total number of M&Ms in the bag (e.g., 23), the ratios might be 8/23 brown; 2/23 red; 6/23 yellow; 2/23 green; 3/23 tan; and 2/23 orange. Although fraction-bar notation is often used, these results may also be expressed using colon notation: 8:23 brown; 2:23 red; 6:23 yellow; 2:23 green; 3:23 tan; and 2:23 orange.

***"All rates are ratios.... So it may help students to work with rates if they think of them in the context of ratios" (Hoffer and Hoffer 1992, p. 317).***

If different measures (e.g., kilograms and days) are compared multiplicatively, the ratio is called a rate. For example, "I produce about 2 kg of garbage per day" can be expressed as 2 kg each day or 2 kg/day. (See Investigation 3, Project Part 2.)

*Operator meaning.* The operator meaning, analogous to a stretching or shrinking process, defines the multiplicative structure of rational numbers, and it is "the most algebraic of the basic ideas" (Kieren 1980, pp. 72–73). For example, "Ask students to describe, illustrate, and provide examples for 'one half of one half.'" (See Investigation 1, Activity 6.) An example of a student's response may be, "A half a dollar is 50 cents. A half of 50 cents is 25 cents, or a quarter. So, half of a half is a quarter."

Mathematics problems involving rational numbers encountered throughout the middle grades have some common underlying structure that reflects one of the four interpretations described above. According to Kieren (1976), to understand rational numbers means that students understand the different interpretations of rational numbers as well as how the different interpretations interrelate. This understanding takes time to develop. *"Understandings are not all-or-nothing affairs, but grow, develop, and expand as a concept is met repeatedly in different contexts and at different levels of abstraction and generality"* (Gibb, Jones, and Junge 1959, p. 39). The most effective way for students to develop these understandings is to present them with opportunities to encounter different meanings within the context of a variety of problem situations.

Important observations about rational numbers may arise out of a discussion of different problem situations. In some situations, different representations depend on the context of the problem. For example, simplifying fractions may not be appropriate when original data are needed. This may occur when analyzing data from the M&M experiment (see Investigation 4), data from the shark simulation (see Investigation 5), or probability experiments. In other situations, one form of notation may be more desirable than other forms. For example, in some cases, fraction notation may be easier to work with than its percent equivalent. This may occur when students analyze data from a recycling project (see Investigation 3). When they construct a circle graph to display data, finding one-fourth of a circle by computing 1/4 of 360 degrees may be easier than finding 25 percent of 360 degrees using its decimal equivalent.

It is no surprise to teachers that the results of national assessments indicate that children have a limited understanding of rational numbers (Kouba et al. 1988), despite the fact that they have rich informal knowledge about fractions in real-world settings (Mack 1990). Using abstract symbols, terminology, and forms of representation without developing meaning on the basis of students' experiences and readiness may cause some of the difficulties that they exhibit. Understanding may be facilitated when teachers not only build on children's informal language but also use appropriate materials to represent the concepts to be developed. The underlying mathematical structure represented by manipulatives or by various situations must be brought forward by allowing children to share their ideas and interpretations through discussing, writing, and reading about rational numbers (Curcio 1990). Investing time to allow students to work in small groups to share their ideas and interpretations about problems involving rational numbers will help them to clarify and evaluate their ideas.

### *What Are Proportions?*

A proportion is a mathematical statement expressing the equality of two ratios. Many mathematics students equate the study of proportions with the "cross multiply" rule (that is, the product of the means equals the product of the extremes). However, the ability to apply this rule is not an accurate indicator of students' understanding of proportions or their ability to recognize a proportional relationship. Understanding proportional relationships requires a deep, complex system of knowledge about a particular quantitative relationship that permeates the world around us. As

***"Middle school students should learn to represent comparison of quantities using ratios in various forms. Ratios should be introduced gradually through discussing the many situations in which they occur naturally. It takes little effort to relate these situations to students' interests" (NCTM 1989, p. 89).***

***It is important to note that some ratios are not rational. For example, the ratio of the circumference of a circle to its diameter is equal to pi, an irrational number.***

***"Teachers should be aware of pupils' viewpoints concerning fractions. The teacher should be like a consulting architect—knowing the ground of a pupil's knowledge and helping the pupil to build a knowledge structure on it" (Kieren and Behr 1985, p. 178).***

***"It is more difficult for students to acquire conceptual understanding once they have learned rote procedures.... Thus, it is essential to focus initial instruction on building conceptual understanding" (Owens and Super 1993, p. 139, citing Wearne and Hiebert).***

a result, students' ability to reason using proportional relationships develops over time.

Several real-world problem settings can be used to illustrate proportional relationships: buying problems (If one-eighth of a cake costs $1.59, find the cost of one-half of the cake; see Investigation 2, Activity 2 Extension); scaling problems (If Carlos needs two jars of paint to cover a two-dimensional representation of his body, how many jars of paint will he need to paint his shadow, which is twice as tall and twice as wide as he is? See Investigation 6, Activity 3); consumption problems (In one month a family generates 120 kg of garbage, 25 percent of which is recyclable paper. How much recyclable paper would you expect the family to generate in one year? See Investigation 3, Project Part 5); and density problems (If thirty-three sharks were tagged and three tagged sharks out of twenty were found during a subsequent sampling, how many sharks would you expect to find in the lagoon? See Investigation 5, Activity 2). As students experience proportional relationships in a variety of contexts, they are able to compare and contrast similarities and differences among these contexts. By explicitly thinking about these relationships and sharing ideas with their peers, learners build a framework for solving problems in which proportional relationships are embedded.

***"Decimals, fractions, ratios, and other multiplicatively based relationships ... eventually play a major role in the development of proportional-reasoning abilities, which, in turn, become the intellectual and mathematical cornerstone of much of what is to come in the secondary school years" (Post 1989, p. 3).***

Presenting meaningful, alternative methods to the "cross multiply" rule strengthens connections between equivalent fractions, unit rates, and proportions. Such methods for solving proportions include the equivalent fraction method and the unit rate method. The examples are taken from Investigation 3, Project Part 5.

*Equivalent fraction method.* What percent of garbage generated in one month is recyclable paper if a family generated 30 kg of paper out of 120 kg of garbage?

$30/120 = 1/4$; $1/4 = ?/100$;
$1/4 \times 25/25 = 25/100$; and
$25/100 = 25\%$

Simplifying 30/120 yields 1/4. Finding a fractional equivalent to 1/4 yields 25/100, which is equivalent to 25 percent.

***With connections to money and with frequent use of this equivalence, students may not need to do the computation.***

*Unit rate method.* In one month a family generated twenty-six kilograms of paper out of sixty-five kilograms of garbage. The following month the family generated thirty-five kilograms of paper, and the ratio of paper to garbage was the same as in the previous month. How much garbage did they generate that month?

65 kg garbage/26 kg paper = 2.5 kg garbage/1 kg paper
2.5 kg garbage/1 kg paper × 35 kg paper = 87.5 kg garbage

This means that there are 2.5 kg of garbage per kg of paper—this is called the *unit rate.*

***"The ability to reason proportionally develops in students throughout grades 5–8. It is of such great importance that it merits whatever time and effort must be expended to assure its careful development. Students need to see many problem situations that can be modeled and then solved through proportional reasoning" (NCTM 1989, p. 82).***

### About This Book

The activities presented in this book are not intended to be a complete rational-numbers-and-proportion curriculum, but rather, they are a collection of ideas that can be integrated into existing middle grades mathematics programs. To focus attention on rational number concepts, traditional computation algorithms for fractions and decimals are noticeably missing. A variety of problems are used to show how rational numbers and proportions are present in many real-world situations. The settings of the investigations have been designed to appeal to middle grades students. On the basis of their knowledge of students' abilities and interests, teachers may need to adjust or modify various aspects of the problems presented.

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

The investigations in this book are categorized in three clusters: Cluster A—Exploring and Extending Rational Number Concepts; Cluster B—Applying Rational Number and Proportion Concepts; and Cluster C—Making Rational Number Connections with Similarity. Cluster A contains two investigations: Happily Handling Halves, an in-depth analysis of the concept one-half, and Benny's Bakery, an exploration of various interpretations of fractions in the context of a bakery theme. Data collection and analysis activities employing ratios, fractions, percents, and proportions are incorporated in Cluster B, which contains three investigations: Garbage and Recycling: The Rational (Number) Approach; The M&M Experiment: A Sweet Approach to Ratio and Proportion; and Sharks in a Box: A Tagging-and-Recapturing Simulation. Cluster C contains two investigations: Trips to Worlds of Shadow Creatures and What in the World Are Fractals? in which rational-number-and-proportion connections are made with similarity.

Each investigation comprises several activities and extensions. Prerequisites for the investigation are identified under Prior Knowledge; the "big" mathematical ideas to be developed are listed under Concepts; and the instructional aims of the activities are stated under Objectives. The teacher notes contain a list of materials needed for each activity, some teaching suggestions, ideas for integrating technology and alternative assessment techniques, and, where appropriate, an answer key. At the end of each investigation are the student activity sheets. All references are listed in the selected bibliography at the end of the book. In addition to a general section, the bibliography contains separate sections for similarity, recycling, and fractal resources.

The suggestions for integrating alternative assessment techniques throughout instruction include diagnostic activities (e.g., Investigation 1), embedded assessments in the form of writing prompts (e.g., Investigation 2), and projects and performance assessments (e.g., Investigation 3). Examples of questions to probe students' understandings are given throughout the activities. These questions can be used to guide the explorations or as possible test items. The activities allow the teacher to assess students on many different levels and for many different purposes. These are examples of how assessment can be viewed as an integral part of instruction. In some cases, samples of students' work are provided (e.g., Investigations 3 and 4).

*It is vital that students have multiple opportunities to express their thoughts and discoveries concerning problems that may or may not have one "correct" answer or that lead into new areas of thought and synthesis.*

Suggestions for integrating technology can be found throughout the book. Calculators without and with fraction capability (e.g., Investigations 1 and 2), the computer (e.g., Investigation 4), and videotape technology (e.g., Investigation 5) are used when appropriate.

### *Moving On*

Just as it was important to recognize the limitations of the whole number system, it is important to recognize the limitations of the rational number system. Eventually, when students need to determine the length of a diagonal in a unit square (i.e., $\sqrt{2}$) or the ratio of the circumference of a circle to its diameter (i.e., $\pi$), they will recognize the need for "new" numbers. Numbers that cannot be expressed as the quotient of two integers are called irrational numbers. Preliminary experiences lay the groundwork for future interest and study of the real number system, which is the union of the set of rational numbers and the set of irrational numbers. The concepts and skills constructed during the middle grades become a foundation on which the more advanced study of the real and complex number systems can develop.

## CLUSTER A: EXPLORING AND EXTENDING RATIONAL NUMBER CONCEPTS

This cluster contains two investigations: Happily Handling Halves and A Trip to Benny's Bakery. Both investigations build on students' informal notions of fractions, decimals, and percents. In the six activities in the first investigation, students focus on various representations of one-half as they relate real-world experiences to short mathematical explorations that lead to creating a bulletin board; translate mathematical expressions into story problems, diagrams, and symbols; test their conjectures using a calculator; and examine the operator meaning of one-half.

A Trip to Benny's Bakery contains six activities. In the first few activities, students employ an area model to represent parts of a whole. They determine how to partition a unit to give the noncongruent fractional parts equal areas. As a challenge in subsequent activities, students are confronted with problems of determining the fractional parts of the whole in which the parts are made up of unequal-sized pieces in the form of tangram and pattern block shapes. The quotient interpretation of fractions, unit cost, and decimal equivalents are highlighted in the activities.

### INVESTIGATION 1: HAPPILY HANDLING HALVES

*Judith S. Zawojewski*

Middle school students typically have a sound understanding of "one-half" from their everyday life experiences:

> "My mathematics homework takes about *half* an hour."
> "About *half* the team went out for ice cream after the game."
> "Mom said she would pay *half* for my new bike as long as I earned the other *half*."
> "Sue's batting average is 0.489. That means she makes a hit about *half* the time she is at bat."
> "I live four-and-a-*half* blocks from school."
> "Alberto's is having a year-end sale on bicycle equipment! Everything is 50 percent off!"

Focusing on "one-half" in a series of short mathematical activities allows students to build on their informal knowledge of rational numbers before they begin a more extensive formal study of fractions, decimals, and percents. The activities are implemented much like short mental mathematics sessions. As a result, students feel a power over many rational number concepts and operations.

Although a form of many of these activities can be done even in the primary grades, the focus for the middle grades is on finding more complex representations of one-half, such as $0.5 billion. This middle grades concept involves connecting large numbers and decimal representations.

The six activities in this section can be used as a diagnostic assessment, helping teachers determine the students' grasp of prerequisites for subsequent activities. Information obtained during diagnostic assessments can be used in making instructional decisions. For each activity, teachers may want to make notes on the types of contributions individual students make to class discussions or on who contributes to the bulletin boards.

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

*Prior knowledge:* Informal, real-world experiences with the concept "one-half"

*Concepts:* One-half of a whole as one-out-of-two equal-sized pieces

*Objectives:*

1. To build an image of one-half of a unit, less than one-half of a unit, and greater than one-half of a unit
2. To partition a unit into two pieces that are congruent and into two pieces that are not congruent but that have equal area
3. To develop facility in mental computation
4. To construct fraction, decimal, and percent equivalences of one-half and "almost" one-half
5. To analyze the relationship between pairs of numbers that yield the fraction one-half
6. To represent the meaning of "half of a half"

### Activity 1: Bulletin Board "Half-Happy"

*Materials:* A bulletin board; newspapers; magazines

*Procedure:* Designate one-half of your bulletin board space to HAPPY HALVES and the other to ALMOST HALVES. Ask students to collect "halves" and "almost halves" from the newspaper, magazines, anecdotes written and illustrated on index cards, and so on. Examples of some creative bulletin board contributions include the following:

> **147 of 310** students at Lincoln School were absent on February 17, during the flu epidemic.
>
> During the first **two quarters** of 1993, Conniwell stock rose 27 points.
>
> The Alpha-Beta Company made profits of $1.**5** million during the 1992 fiscal year.
>
> I cut a small pizza into thirds to share with my brother and sister. Since my brother decided he did not want any, my sister and I shared it by taking **1 and 1/2 thirds** each.

Follow-up discussions can introduce and use the terms *fraction, decimal, percent, ratio,* and *complex fraction* to describe the many different types of one-half. The ALMOST HALVES bulletin board provides opportunities to talk about "greater than a half" and "less than a half."

*Diagnostic Assessment:* Have pairs of students use a calculator to change the "almost halves" to decimals. Ask the pairs to develop rules or guidelines for deciding when a fraction is "almost half" or "not almost half." You can use the students' guidelines to assess their understanding of comparing, ordering, and "closeness" and their concepts of decimals and fractions. For example, if a student indicates that 0.049 is "almost half," you have information to use in planning your instruction for at least this pair of students.

### Activity 2: Creative Halves

*Materials:* Overhead transparency or diagram on the chalkboard of the yin-yang symbol (fig. 1.1); four to six index cards for each student; rulers; pencils

*Procedure:* Illustrate a variety of "creative" visual halves, such as the yin-yang symbol shown in figure 1.1.

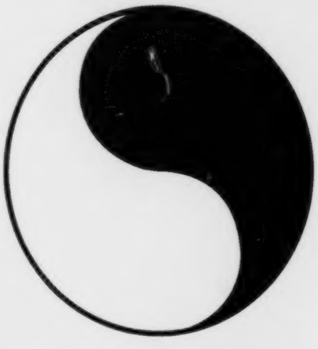


***Fig. 1.1. The yin-yang symbol***

Give each student four to six index cards, a ruler, and a pencil. Ask them to develop their own "creative halves." Some possible creations are shown in figure 1.2.

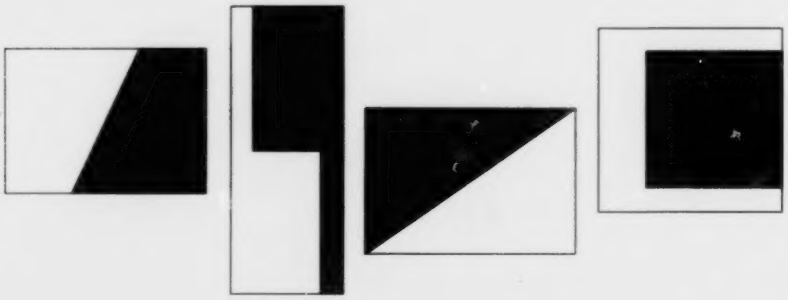


***Fig. 1.2. Creative halves***

*Writing Prompt:* Present students with the following situation: Chris said that all the shaded halves in the rectangles of figure 1.2 are equivalent. Pat disagreed, saying they were different shapes. With whom do you agree? Explain your position. (*Note:* Each shaded region indicates half the area of the original rectangle, so they are indeed equivalent.)

*Extensions:* See Investigation 2: A Trip to Benny's Bakery, Activities 1 and 2. In the context of cutting rectangular sheet cakes into eighths and fourths so that the pieces contain the same amount of cake, students are challenged to represent equal parts of a whole with noncongruent pieces. For other ideas related to noncongruent parts of a whole with equal areas, see Corwin, Russell, and Tierney (1990).

***Activity 3: Heady Halves***

***Communication: This particular situation can be rephrased, "How many halves are in three wholes?" which is phrased like whole number language.***

Middle school students are quite capable of dealing with "halves" mentally when problems are presented to them orally in meaningful contexts. They are able to create and use their own visual representations as needed. This activity contains a series of exercises presented in a suggested sequence. A discussion of each type of item may last a few days before the class moves on to another. It is always useful to introduce the example by asking students to make up word problems or situations that would require the particular operation. For example, 3 divided by 1/2 could be "realized" by the following illustration: If I have three sandwiches and I plan to give each child half a sandwich, how many children can be given one-half of a sandwich? Mixed review is essential. Previously covered examples can be mixed into the short daily mental mathematics sessions, keeping students thinking about the meaning of halves and the various operations.

*Materials:* None

*Procedure:* Present the following questions to students orally and ask them to think of a situation in which the operation would be used. Have them solve the problem mentally.

"What is 1/2 plus 2 1/2?"
"What is 3 minus 1/2?"
"What is one-half of 24?"
"What is 50 percent of 24?" (Relate this to the previous question.)
"How many halves fit into three whole units?"

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

"What is 3 divided by 1/2?" (At first, relate this to the question about sharing half-sandwiches.)

"Is one-half of 47 greater than or less than 20? 30? 25?"

"Is 55 divided by 1/2 greater than or less than 50? 100?"

*Connections:* Link the oral question to the written symbolic question by asking students to record 1/2 × 24 and to tell stories about "What is one-half of 24?"

*Writing Prompt:* Give students a question, such as "What is 3 divided by 1/2?" and ask them to write a story, draw a picture, and express the mathematical statement symbolically. The teacher can use the students' work to assess which representational modes are well connected.

**Activity 4: Here's Looking at Halves**

Students can begin investigating equivalence among fractions, decimals, and percents by using concrete representations.

*Materials:* Two copies of Activity Sheet 1.1 for each student; scissors

*Procedure:* 1. Distribute a copy of Activity Sheet 1.1 to each student. Have them cut around the perimeter of the "tenths" and the "hundredths" squares. Review the meaning of "tenths" and "hundredths" by asking the students to visualize the whole square unit being cut up into the smaller pieces.

2. Ask students to work in pairs as they fold into two halves the square partitioned into tenths. Ask them to shade one-half of both sides of the paper; one-half can then be viewed both with respect to the number of equal parts of the whole present in one-half of the whole and without any reference to the equal parts. Have them record their halves in as many ways as they can think of, later sharing their ideas with the class. Encourage students to think creatively by reminding them that squares have more than two lines of symmetry (i.e., diagonal as well as vertical and horizontal). Some possible responses are presented in figure 1.3.

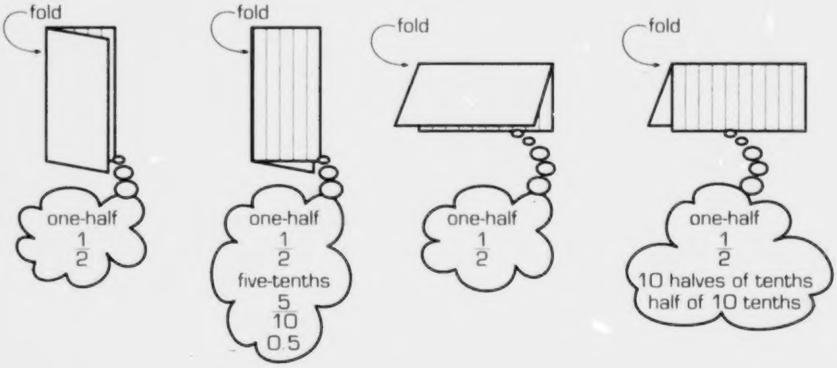

fold
fold
fold
fold
one-half 1/2
one-half 1/2 five-tenths 5/10 0.5
one-half 1/2
one-half 1/2 10 halves of tenths half of 10 tenths

**Fig. 1.3**

3. Ask students to work in pairs as they fold into two halves the square partitioned into hundredths. Again ask them to shade one-half of both sides of the paper and to record their halves in as many ways as they can. Each pair should share their own way of thinking about halves with the class. Some possible responses are presented in figure 1.4.

***Technology: Using calculators with fraction capabilities, students can directly translate between decimals and fraction representation and use the $\boxed{\text{Simp}}$ function to explore equivalence among fractions. Students can be asked to compare the $\boxed{x \rightleftarrows y}$ key's procedure to the division procedure they have discovered in this activity. How are the keystroking procedures alike and different? How are the results alike and different? How are these two keystroking procedures related?***

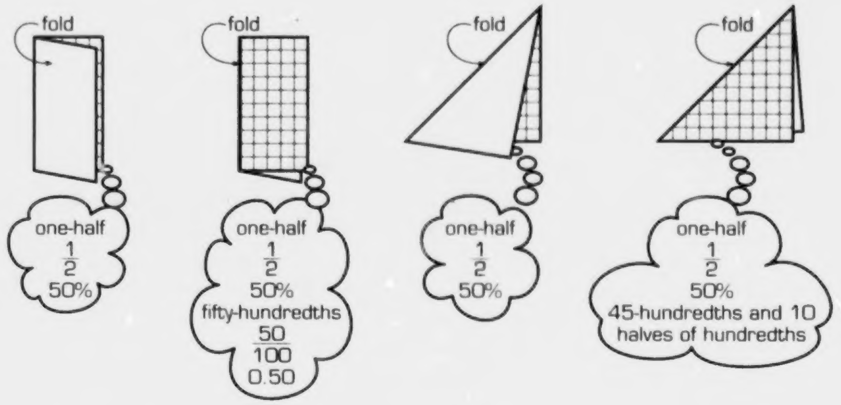


**Fig. 1.4**

*Extension:* Other physical representations, such as number lines, parts of sets, and so on, can be investigated in a similar fashion.

***Activity 5: Calculator Conjectures***

*Materials:* Blank bulletin board labeled CALCULATOR CONJECTURES; a calculator without fraction capability for each pair of students; index cards; Post-it notes

*Procedure:* 1. Ask students to describe how they can fold halves with a piece of paper. (The whole sheet is divided into two equal parts.)

2. Relate this to division: 1 ÷ 2. Ask students to do the same on the calculator. Ask them if the results make sense. The 0.5 in the display should make sense, being confirmed in their informal knowledge as well as by the Here's Looking at Halves activity described above.

3. Ask students to find other ways to make the calculator have an "answer" of one-half. Each pair of students can present findings to the class on the chalkboard, which provides an opportunity to represent the operations symbolically. Their classmates can try each example and discuss why it does or does not work. Some illustrations follow:

| | *Calculator key strokes* |
|---|---|
| $2 \div 4 =$ | $\boxed{2}\ \boxed{\div}\ \boxed{4}\ \boxed{=}$ |
| $5 - 4.5 =$ | $\boxed{5}\ \boxed{-}\ \boxed{4}\ \boxed{.}\ \boxed{5}\ \boxed{=}$ |
| $1.5 \div 3 =$ | $\boxed{1}\ \boxed{.}\ \boxed{5}\ \boxed{\div}\ \boxed{3}\ \boxed{=}$ |
| $0.25 \times 2 =$ | $\boxed{.}\ \boxed{2}\ \boxed{5}\ \boxed{\times}\ \boxed{2}\ \boxed{=}$ |

4. After students begin to see patterns for generating further examples, they can be asked to generate rules to get the calculator to answer "0.5." For example, "a number divided by its double will always be 0.5." These conjectures are written on an index card, posted on the CALCULATOR CONJECTURES board, and tested by classmates. As classmates test each conjecture, they sign the card if they agree that the rule holds true. If they disagree, they should write a counterexample on a Post-it note, sign the counterexample, and attach it to the conjecture card. A

***Reasoning about Proof: Some middle school students accept conjectures as "proved" on the basis of a finite set of examples that have "worked." Opportunities to discuss the nature of proof arise naturally when students find counterexamples: They have "proved" the conjecture does not hold for all cases. At this point it is productive to explore questions regarding whether they have "proved" the conjectures they have accepted up to this point. The notion of deductive proof is nicely introduced when students realize it is often impossible to test every case of a conjecture.***

conjecture card is laminated once everyone in the class agrees that it should remain (including the teacher, who is the last to sign it).

*Assessment:* Each student can be required to respond to two conjectures of her or his choice. In their journals, they indicate whether they agree or disagree with the conjectures and explain their reason(s).

***Activity 6: Half of a Half***

*Materials:* Calculators; paper and pencils

*Procedure:* 1. Ask students to describe, illustrate, and provide examples for "one-half of one-half." Encourage a variety of ways to find a half of a half. Some examples follow:

> "Take a paper and fold it in half. Then fold it in half again. When unfolded, you can see the resulting regions are each 1/4, one-fourth, or one quarter."
>
> "Take the hundredth square and fold it in half, and then fold the half into half again. On the side with hundredths showing, you can see that each of the regions is 0.25, 25 hundredths, or 25 percent."
>
> "A half of a dollar is 50 cents. A half of 50 cents is 25 cents, or a quarter. So, a half of a half is a quarter."
>
> "On the calculator we found that you could find half of a number by multiplying it by 0.5. So, if you take 0.5 × 0.5, you get 0.25. So, a half of a half is 0.25."

***If students have access to calculators with fraction capability (e.g., the TI Math Explorer), they could take the decimal result, 0.25, use the decimal-to-fraction key, F↔D, to convert it to 25/100, and then, with the Simp key, simplify it to 1/4.***

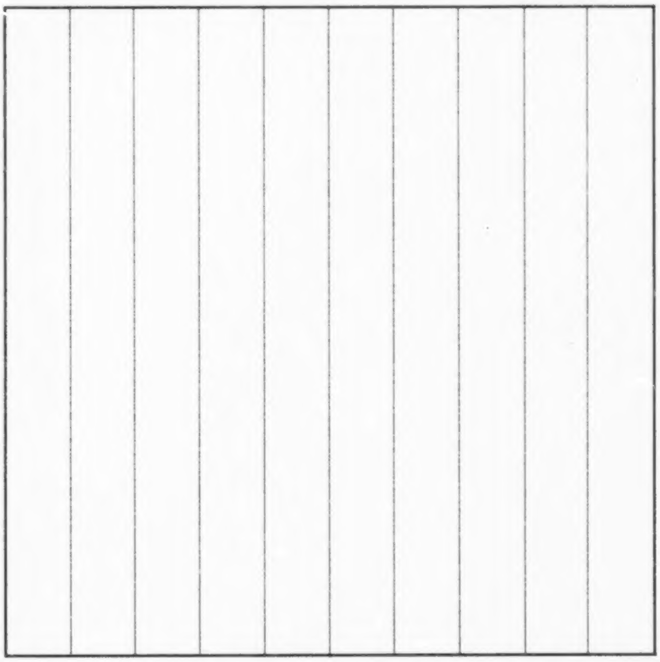
*Suggestions for Extensions:* In addition to the activities described here, estimation with 1/2, 0.5, 0.50, 50 percent, and so on, can be included in the experiences with halves. Many of these types of activities are described in *Developing Number Sense in the Middle Grades* (Reys 1991). Moving into mental arithmetic with fourths, quarters, 25 percent, and so on, is also natural at this point. Students can easily visualize the sum and difference of 1/2 and 1/4. Additionally, teachers and students who have started investigating many aspects of one-half often cannot stop finding further explorations and perspectives of their own. Let your mind wander to see how many different topics in rational numbers you can uncover by letting students' ideas and insights lead the way.

## CLOSING THOUGHTS

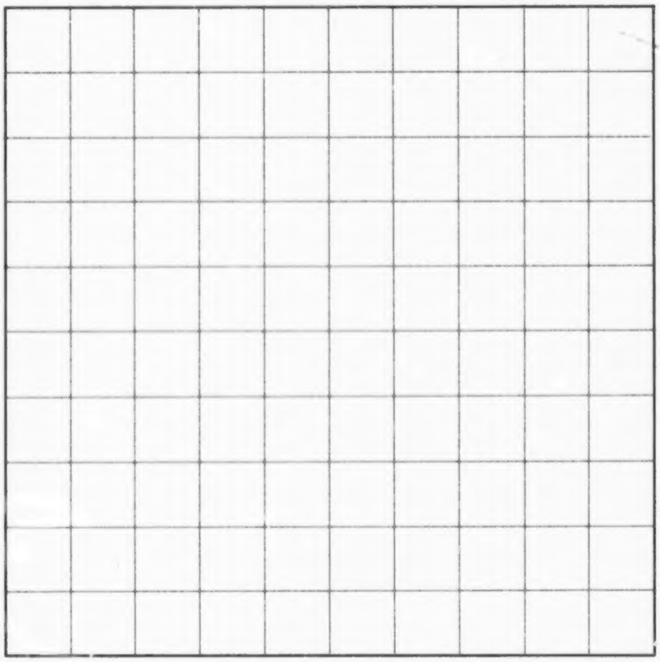
Since students are simultaneously exposed to fractions, decimals, and ratios in everyday life, it is artificial to sequence the introduction of fractions, then decimals, and then percents in strict order. Rather, by taking clues from the informal knowledge that students have already acquired, teachers can provide opportunities for students to build on and formalize the concepts and ideas they have already developed. The Happily Handling Halves activities described in this chapter are only samples of how we can help students use their existing knowledge to develop more formal understandings. Once teachers and students become accustomed to letting their own knowledge base guide instruction, ideas for learning about rational numbers in a sense-making mode will explode. Both students and teachers become excited about playing with these numbers and constantly relating them to their own experiences.

Name ______________________________ Date ______________

**HERE'S LOOKING AT HALVES**



Directions: Cut around the *perimeter* of the "tenths" and "hundredths" squares. Fold in half and shade both the lined and unlined sides of the squares to represent one-half.



**ACTIVITY SHEET 1.1**

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

## INVESTIGATION 2: A TRIP TO BENNY'S BAKERY

*Nadine S. Bezuk, Barbara E. Armstrong, Heidi Janzen, and Steven T. Klass*

The six activities in this investigation center on situations involving rational numbers and proportions that students encounter during a fictitious trip to a bakery. These activities involve several important concepts of rational numbers and proportions, including partitioning a unit into equal parts, the quotient interpretation of fractions, the area model of fractions, determining fractional parts of a unit not cut into equal-sized pieces, equivalence, unit prices, and multiplication of fractions; all of these are presented in a real-world setting. The activities emerge from problems posed by Benny.

| | |
|---|---|
| *Prior knowledge:* | Partitioning a whole into equal-sized pieces; naming fractional parts of a whole; rounding to the nearest cent; multiplication and division |
| *Concepts:* | Area model of fractions; partitioning a whole into unequal-sized pieces; quotient interpretation of fractions; equivalent fractions; decimals; unit rates; proportion |

*Objectives:*

1. To represent parts of a whole using an area interpretation of fractions
2. To determine the fractional part of a whole when parts are not cut into equal-sized pieces
3. To develop an understanding of the quotient interpretation of fractions
4. To find the unit cost of items that are part of a set
5. To determine the relationship among parts of a whole that are unequal-sized pieces
6. To express fractional parts of a whole as decimal equivalents

### Activity 1: Customers Cut the Cake

*The Problem:* Each day Benny makes several rectangular sheet cakes, which he cuts into eighths. He sells 1/8 of a sheet cake for $1.59. As part of a new promotional campaign for his store, he wants to cut his sheet cakes into eighths a different way each day. Customers who suggest a new way to cut the cakes into eighths win a free piece of cake each day for a week. What are some of the different ways to cut the cake?

**This activity gives students opportunities to represent parts of a whole by using an area model of fractions. Many students know that fractions often refer to equal-sized parts of a unit, but they frequently overgeneralize and believe that the pieces have to be congruent rather than merely to contain the same area.**

*Materials:* Activity Sheet 2.1 (or scrap paper); scissors; Overhead Master 2.1 (for overhead projector); projector

*Procedure:* 1. Introduce the problem by discussing the situation at Benny's Bakery. Consider the following questions for discussion:

> What is the shape of Benny's cakes? (*Rectangular cakes*)
>
> What are the restrictions on the ways the cakes can be cut? (*Pieces must contain the same amount of cake; they do not have to be the same shape.*)

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

How can we verify that pieces that are not the same shape contain the same amount of cake? (*Cut the pieces into smaller parts and lay one on top of the other.*)

2. Encourage students to solve the problem in pairs or in small groups by using Activity Sheet 2.1. Alternatively, students can use rectangular pieces of paper to model the cakes, sketching the shapes or cutting the paper into eight pieces and verifying the equivalence of the pieces by cutting and overlapping. Some students may have a limited view and think that all the cuts must be parallel to one side of the rectangle. Challenge them to think of other ways to make the cuts.

3. Have students place their designs on the chalkboard or the overhead projector. Ask the students to decide which designs are the same and which are different. Examples of some diagrams are in figure 2.1. Have the students discuss whether the two rectangles in figure 2.1 are cut differently.

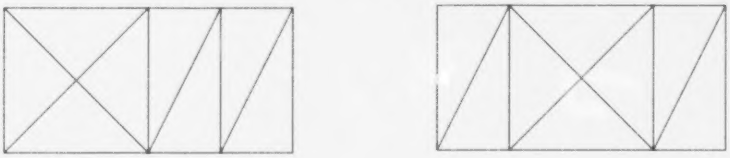


***Fig. 2.1***

What factors should be considered when deciding whether the two designs are different? (*The number of pieces, the equivalence of the pieces, and whether the location of the pieces makes a difference should be considered.*) The two rectangles in figure 2.1 contain the same eight pieces, but the pieces are arranged differently. The students can decide whether they want to consider these as two different arrangements.

4. Another way that must be considered is using cuts that are curves or combinations of line segments, such as the examples in figure 2.2.

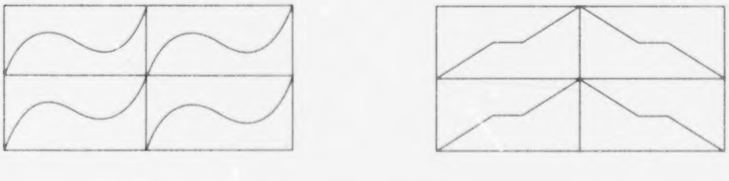


***Fig. 2.2***

The equivalence of shapes formed by cuts that are curves is difficult to determine but is a good investigation in itself. The equivalence of shapes that are formed by cuts that are combinations of straight line segments is easier to determine. Including these types of shapes, however, again greatly increases the number of possibilities. By interacting with students, decide which designs to include in the final count.

5. Ask each group to choose one design and either to post it on the chalkboard or bulletin board or to draw it on the overhead projector. Students in each group should be prepared to explain how they know that their method shows eighths. One way to verify that a solution does in fact result in eighths is to cut up the individual pieces further and lay them on top of each other to verify the equivalence of their areas. The

***The value of this activity is not in determining the final "right" answer but rather is in the process of determining and considering which factors are relevant and how various factors affect the situation. If all types of factors are considered, there are an infinite number of ways to cut the cake into eight equal pieces.***

resulting discussion is very important and will yield a great deal of rich assessment information regarding the students' understanding of part-whole representations using an area model of fractions. As a follow-up activity, teachers may choose to discuss with students why or why not each cake on Overhead Master 2.1 is cut into eighths. An alternative approach would be to give one example to each group, making the students responsible for explaining why it shows eighths.

*Summary:* Have students display, discuss, and validate their designs. Some of the possible ways to cut sheet cakes into eighths are shown in figure 2.3. Students will probably first discover the two ways at the top of the figure, which have cuts that are parallel to the sides of the cake. They are partitioned by using the more commonly used part-whole interpretation of fractions, in which a whole is partitioned into a certain number of *congruent* parts. Most of the remaining rectangles illustrate parts of a whole in which the *area of the shapes is the same* but the pieces are not congruent. An exploration focusing on noncongruent representations of one-half can be found in Investigation 1, Activity 2. For other ideas regarding noncongruent fractional parts of a whole, see Corwin et al. (1990).

***Be open to such unusual solutions as weighing the various pieces and using ratios of weights rather than ratios of areas. This method would lead to different, though equally valid mathematical applications.***

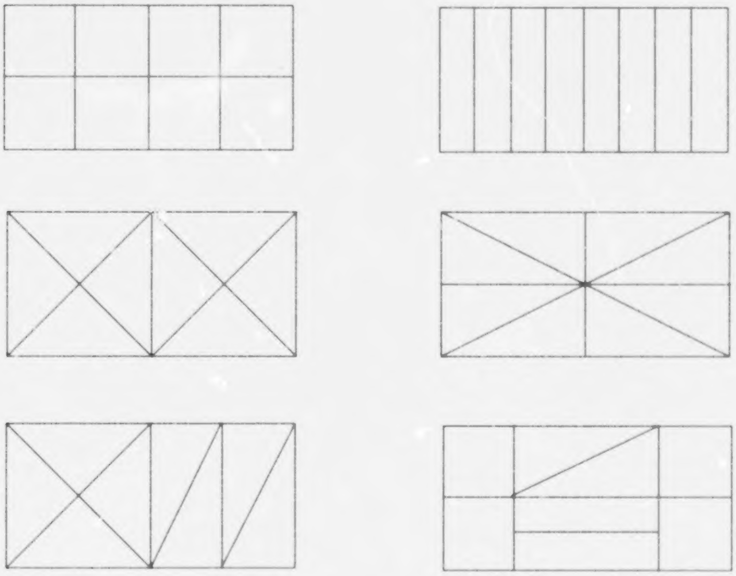


***Fig. 2.3***

### *Activity 2: You Can Eat Your Cake and Have It, Too!*

*Problem:* Benny is conducting a second contest, this time for his employees. As part of a new promotional campaign for his store, each day he wants to feature sheet cakes that have been cut into four pieces in a different way. The pieces do not have to be equal for this promotion. Benny has challenged his employees to suggest interesting ways to cut the cakes into four pieces. The employees must also determine the price for each piece. Benny sells 1/8 of a sheet cake for $1.59. What are some of the different ways the cakes can be cut, and how much should each piece cost?

*Materials:* Activity Sheet 2.1 (or paper); scissors; Overhead Master 2.2 (for overhead projector); projector; calculators with fraction capability (optional)

***Instead of focusing on making equal-sized pieces as in Activity 1, this activity explores determining the fractional parts and cost of pieces when a unit is cut into four unequal parts.***

*Procedure:* 1. Introduce the problem by discussing this new situation at Benny's Bakery. Include the following questions:

> What are the restrictions on the ways the cakes are cut? [*Each cake must be cut into four pieces—not necessarily equal-sized pieces.*]
>
> How can we determine the fractional parts of the pieces we cut? [*We can use equivalences we know, such as 2/4 is equivalent to 1/2, to find the value of each part, if we partition by finding parts of parts.*]
>
> What will happen if we just cut four pieces at random? [*It will be difficult to determine the size of each piece.*]

***Once again, this problem could be solved by weighing.***

2. Encourage students to solve the problem in pairs or small groups by using Activity Sheet 2.1 or rectangular paper to model the cakes. They sketch the pieces or cut the rectangles into four pieces and determine the value of each piece. One way to find the value of each piece is to add partitioning lines so that the whole is partitioned into equal-sized pieces. Students may remember some of the ways they cut the cakes into eighths in Activity 1, which may help them.

3. Have students share and discuss their designs. The rectangles in figure 2.4 show a few possible ways to cut the rectangular cakes into four parts. The pieces have been labeled to show the fraction of the cake they represent.

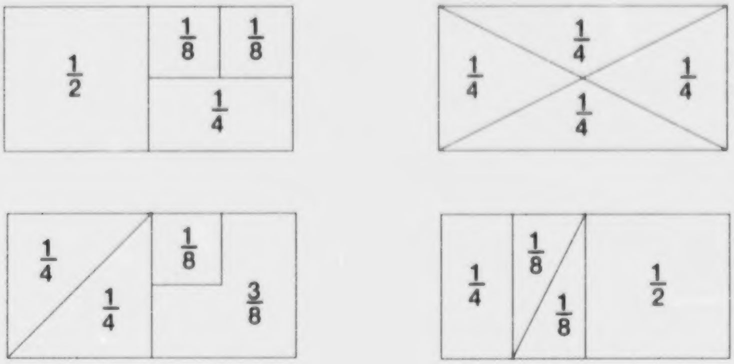


***Fig. 2.4***

***Reasoning: Students verify their thinking.***

4. Students must then find the cost for each piece, if 1/8 of a cake costs $1.59. What should the total cost of one whole cake be? [*If 1/8 of a cake costs $1.59, then a whole cake should cost eight times as much, or $12.72.*]

*Summary:* Challenge students to explain and verify their solutions.

*Extension:* Students find the cost of various-sized pieces, given that 1/8 of a cake costs $1.59 and a whole cake costs $12.72. The following table is a sample. (Tables may include other fractional parts and need not be limited to eighths, fourths, and halves.) Students may wish to use calculators with fraction capability to help them find the various prices. Using calculators may help students focus on the reasonableness of their solutions rather than on the calculations.

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

| *Size* | *Cost* |
|---|---|
| 1/8 | $ 1.59 |
| 2/8 = 1/4 | 3.18 |
| 3/8 | 4.77 |
| 4/8 = 1/2 | 6.36 |
| 5/8 | 7.95 |
| 6/8 = 3/4 | 9.54 |
| 7/8 | 11.13 |
| 8/8 = 1 | 12.72 |

*Embedded Assessment:* Teachers can assess students' understanding of naming fractions and fractional equivalents by focusing on how students solve this problem. For example, are students able to recognize that 4/8 is equivalent to 1/2? Do students use their knowledge that 6/8 is equal to six one-eighths to help them find the cost of 6/8 of a cake (i.e., since 1/8 of a cake costs $1.59, 6/8 of a cake costs six times as much, or $9.54)?

**Activity 3: That's the Way the Cookie Crumbles!**

*Problem:* You bought a baker's dozen (13) of cookies that you want to share equally with your family. How many cookies will each person get?

*Materials:* Activity Sheet 2.2 (or paper); scissors (optional)

*Procedure:* 1. Ask students to compare this problem with the one posed in Activity 1. Give them time to think about the similarities and differences between this problem and the problem posed in Activity 1. (*This problem is similar to the first problem in Benny's Bakery, that of cutting rectangular sheet cakes into eighths in different ways, because they both involve partitioning a unit into parts. This problem is different in several ways: The whole or unit in this case is a group of thirteen cookies; the problem does not specify exactly into how many pieces to cut each cookie or how many people are sharing the cookies; the problem has different solutions for students who have different-sized families; and this problem involves a different interpretation of rational numbers—the quotient interpretation. The quotient interpretation refers to the fact that in this problem, in which thirteen cookies are being shared by n people (n is the number of people in the family), the number of cookies each person receives is the quotient, 13/n. This interpretation is most commonly found in applications (Lay 1982), and it is of instructional importance.*)

2. Consider grouping students according to the number of people in their families. After each group has solved the problem, share the solution processes with the whole class. Have each student complete a table similar to the sample below.

| Number of people in the family | Number of cookies for each family member |
|---|---|
| 2 | 13/2, or 6 1/2 |
| 3 | 13/3, or 4 1/3 |
| 4 | 13/4, or 3 1/4 |
| 5 | 13/5, or 2 3/5 |
| 6 | 13/6, or 2 1/6 |
| ⋮ | ⋮ |
| *n* | 13/*n* |

The table should include all the different-sized families of students in the class and also contain a few other family sizes, including one or two families that are larger than the largest family in the class. Students should begin completing the table by including solutions from the groups in the class and then working on solving the problems for other family sizes.

Discuss the solutions, focusing on the patterns students see in the number of cookies for each person as the size of the families changes. The goal for students is to generalize that the number of cookies for each person is equal to the number of cookies divided by the number of people sharing them.

*Summary:* Ask students to write a rule that represents the number of cookies each person receives if thirteen cookies are shared by $n$ people ($n$ is the number of people in one's family). [*The number of cookies each person receives is the quotient,* $13/n$.]

*Writing Prompt:* Ask students to describe the way they solved the problem in their groups (e.g., by drawing a picture, using long division, and so on) so that a student in another class would understand what problem was solved and how it was solved.

*Extension:* How many cookies would each person get if twenty cookies were shared by three people? [*20/3, or 6 2/3 cookies for each person*] By eight people? [*20/8, or 2 1/2 cookies for each person*] By $x$ people? [$20/x$ *cookies for each person*] What is a rule for finding the number of cookies each person will get if $a$ people share $b$ cookies? [$b/a$ *cookies for each person*]

***In solving this problem, students use their understanding of multiplication and division, fractions, and proportional reasoning.***

**Activity 4: How Much Is That Cupcake in the Window?**

*Problem:* Benny sells four cupcakes for $1.25. He would like to have a table for new employees that tells them how much to charge when people want to buy different numbers of cupcakes. Benny says that sometimes people buy only one cupcake and that people hardly ever buy more than two dozen cupcakes.

*Materials:* Calculators (optional)

*Procedure:* 1. Ask students how they could help Benny's employees. [*The problem presents three numbers and asks students to find the fourth; that is, if four cupcakes cost $1.25, how much will one cupcake cost? Eight cupcakes? A dozen cupcakes? Students can solve these problems as they would a standard proportional reasoning problem, that is,* $a/b = c/d$, *in which* $a$, $b$, *and* $c$ *are given, and the student is asked to find* $d$. *They could also approach the problem by using division to calculate the cost of one cupcake and then multiplying to find the cost of* $n$ *cupcakes. Students must either use rational numbers—fractions or decimals of cents (e.g., $0.215 or $0.21 1/2)—or round off their answers, which introduces rounding error. Two methods that may be used to solve problems involving proportions are discussed in the Introduction, p. 4.*]

2. Depending on students' responses and how difficult they find the problem, consider making the problem simpler by changing the price to a multiple of the number of cupcakes. For example, four cupcakes cost $1.20. How much is one cupcake?

*Summary:* Have students describe how they solved the problem. Did any students use the unit rate strategy? Have students discuss whether any problems were easier or more difficult to solve than others, and why.

*Embedded Assessment:* Teachers can use this activity to assess students' understandings of proportional reasoning situations, their division skills, and their understanding of how to deal with remainders in a real-world situation.

***Activity 5: There Are Pattern Blocks in My Cake!***

*Problem:* Benny wants to have a "cake of the week," called the Pattern Block Cake, and offer it in two sizes: family size and personal size. Each Pattern Block Cake (see fig. 2.5) is formed from smaller cakes of different flavors, with each flavor a different shape. Prices are determined by the size of the cake, not by its flavor. The Pattern Block Cake will consist of chocolate cake cut into triangles, marble cake cut into rhombi, spice cake cut into trapezoids, and white cake cut into hexagons. Like pattern blocks, the different-shaped pieces of cake are related to each other (i.e., six triangles are equivalent to one hexagon, three triangles are equivalent to one trapezoid, and two triangles are equivalent to one rhombus).

Benny needs help in determining what he should charge for the whole family-size cake and for each different piece. If each small triangular piece costs $1.00, how much will the other pieces cost? How much will the whole cake cost?

*Materials:* Pattern blocks; Activity Sheet 2.3; Activity Sheet 2.4

*Procedure:* 1. Let students construct a model of the cake with pattern blocks. This process will facilitate their making trades to find the relationships between the fractional parts.

2. Challenge students to find relationships between and among the different-sized pieces in the Pattern Block Cake. Ask them to find the monetary value of the different pieces (see fig. 2.6) and the maximum possible number of triangles in each row of the cake (see fig. 2.7). Finding the maximum possible number of triangles in each row presents an opportunity to discuss triangular numbers, that is, numbers that can be arranged to form a triangle. (For a further discussion of patterns with figurate numbers, see Phillips [1991].)

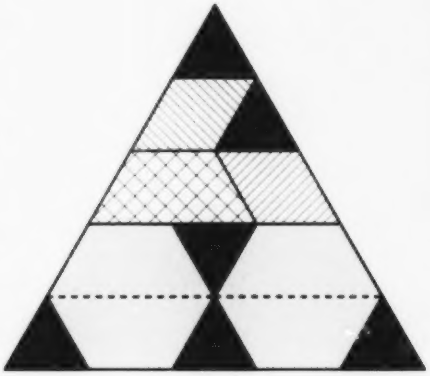


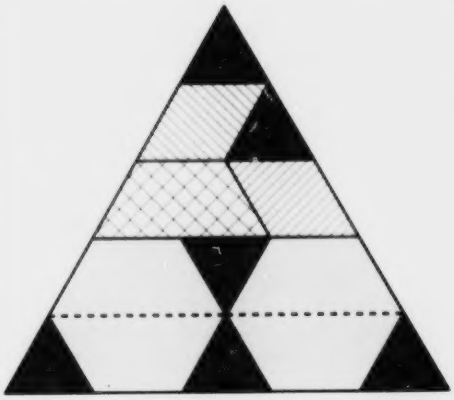
***Fig. 2.5***

*Summary:* Let students share their findings and discuss their strategies. Highlight the relationships among the unequal-sized pieces and their monetary equivalents. The total cost for a whole Pattern Block Cake should be $25.00, the sum of the total number of triangles (or their equivalent) in each row of the cake.

*Extension 1:* If Benny wanted to make the cake bigger and add another row to the bottom of the cake, what might that row look like? What is the new cost for the whole cake?

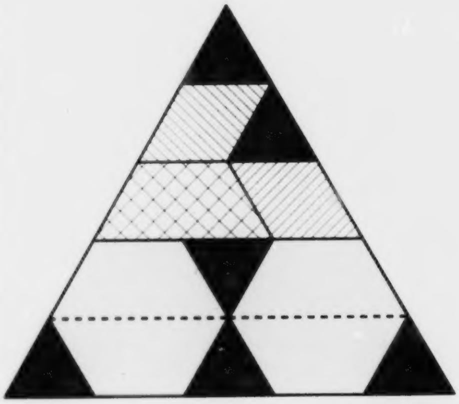
Some possibilities for the next row can be found in figure 2.8, diagrams 1 and 2. Diagram 3 shows a third possibility if two rows are added to the original cake. The patterns discovered in Activity 5 will be helpful in

***Many students solve proportional reasoning problems by using the unit rate strategy. Students using this strategy might solve the problem in Activity 4 in the following way: Since four cupcakes cost $1.25, the cost of one cupcake is $1.25/4, or $0.3125 (rounded to $0.32 because we are dealing with money), which is the unit price (or unit rate). The unit price is powerful because it can be used to find the price for any number of items. So two cupcakes would cost 2 × $0.32, or $0.64, and so on. This is also a good opportunity to discuss how to deal with remainders and rounding in a real-world situation.***



| Name of Piece | Cost |
|---|---|
| Triangle | $1 |
| Rhombus | $2 |
| Trapezoid | $3 |
| Hexagon | $6 |

*Fig. 2.6*



| Maximum Possible Number of Triangles in Each Row |
|---|
| 1 |
| 3 |
| 5 |
| 7 |
| 9 |

*Fig. 2.7*

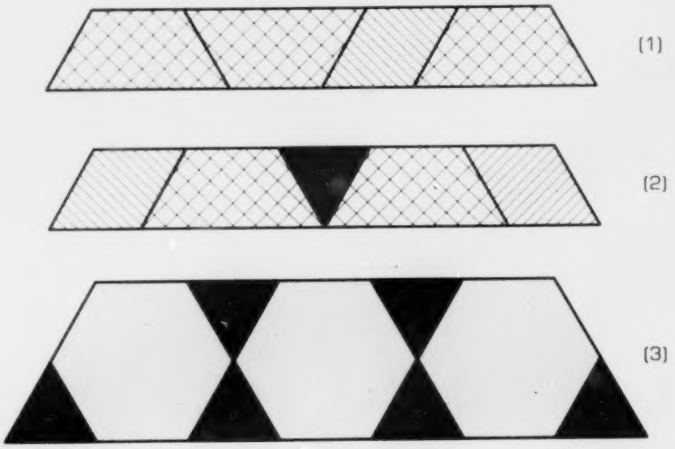


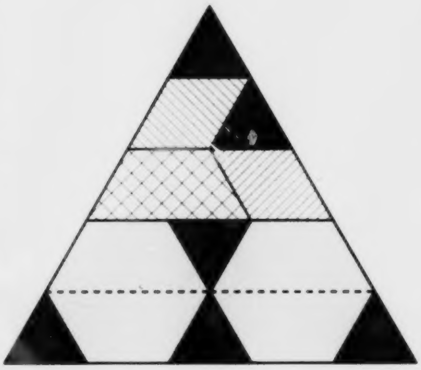
*Fig. 2.8*

solving this problem. Teachers may ask students to describe the patterns in the table, perhaps writing a formula representing the pattern of the number of triangles in each row.

*Extension 2:* A whole personal Pattern Block Cake costs $1.00. How much will the pieces cost?

This problem is more difficult than that initially posed in Activity 5. A personal Pattern Block Cake is a smaller cake similar in shape to the family Pattern Block Cake. It contains the same shapes in the same arrangement, but each shape is smaller.

In this problem, the unit is defined as the whole cake rather than a triangular piece, as in Activity 5. This is another opportunity for teachers to examine students' understanding of equivalent fractions and of identifying fractions when the unit is partitioned into unequal pieces (see fig. 2.9).



| Name of Piece | Part of the Whole | Cost |
|---|---|---|
| Triangle | $\frac{1}{25}$ | $0.04 |
| Rhombus | $\frac{2}{25}$ | $0.08 |
| Trapezoid | $\frac{3}{25}$ | $0.12 |
| Hexagon | $\frac{6}{25}$ | $0.24 |

***Fig. 2.9***

Ask students how they could check their results for accuracy. [*The total cost of the whole cake should be $1.00.*]

*Extension 3:* If Benny changes the price of each triangular piece of the family Pattern Block Cake to $0.89, how much will the other pieces cost? How much will the whole family Pattern Block Cake cost?

*Embedded Assessment:* Teachers can use this activity to assess students' understandings of equivalent fractions and their ability to multiply decimals.

### *Activity 6: There Are Tangrams in My Cake!*

*Problem:* Benny wants to have another "cake of the week" be a Tangram Cake. Each Tangram Cake will consist of a square cake cut into the seven tangram pieces (see fig. 2.10). Benny needs help in determining what he should charge for each different piece, since some customers may want to buy only a part of the Tangram Cake. If each Tangram Cake costs $1.00, how much will each piece cost?

*Materials:* Overhead Master 2.3; scissors (optional); Activity Sheet 2.5; calculators (optional)

*Procedure:* 1. Students can fold and cut a piece of paper into tangram pieces (see Russell and Bologna 1982) or cut apart a Tangram Cake

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

Photo by Gregory Grambo

***Seventh graders exploring with a tangram cake***

using activity sheet 2.5. They can use these pieces as manipulatives to help them determine the fractional part (see fig. 2.10).

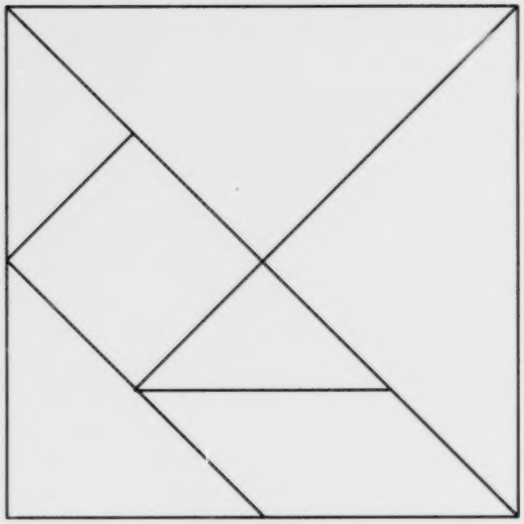
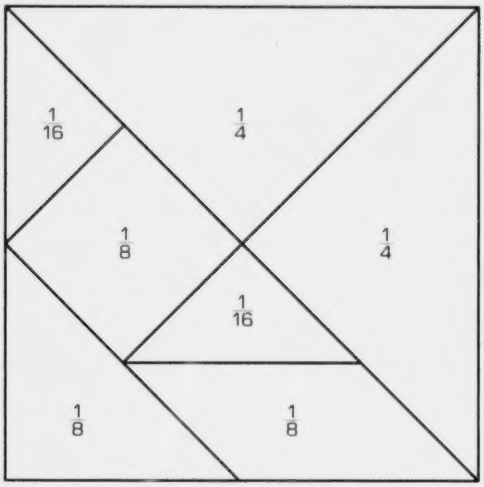


***Fig. 2.10***

2. Have students identify the fractional value and cost of each piece of the Tangram Cake, if the whole cake is the unit and it costs $1.00 (see fig. 2.11).

*Summary:* Have students explain their strategies for solving the problem and their solutions. Have them explain the relationship between and among the tangram pieces.

*Extension 1:* Benny wants to make large Tangram Cakes that cost $10.00 each. How much should he charge for each piece of the larger tangram cake? What if he made a giant Tangram Cake that costs $1 000.00—how much should each piece cost?



| Name of Piece | Fraction of Cake | Cost |
|---|---|---|
| Large triangle | 1/4 | $0.25 |
| Medium triangle | 1/8 | $0.12 1/2 |
| Small triangle | 1/16 | $0.06 1/4 |
| Square | 1/8 | $0.12 1/2 |
| Parallelogram | 1/8 | $0.12 1/2 |

***Fig. 2.11***

This extension provides students with an opportunity to examine place-value issues related to decimals. For example, when the Tangram Cake costs $1.00, the medium triangle (which is 1/8 of the cake) costs $0.12 1/2. If the large Tangram Cake costs $10.00, which is ten times as much as the regular Tangram Cake, each piece should cost ten times as much. The medium triangle will now cost ten times $0.12 1/2, or $1.25. Each piece will still have the same relationship to the other pieces, however.

***This problem is quite rich because it forces students to examine and apply their understandings of the concept of fraction, particularly since the unit as provided is not partitioned into equal-sized parts. Students must examine the relation between various pieces to determine the fractional parts of the unit. For example, what part of the whole are the two large triangles? How many small triangles fit in one large triangle? What part of the whole is the small triangle?***

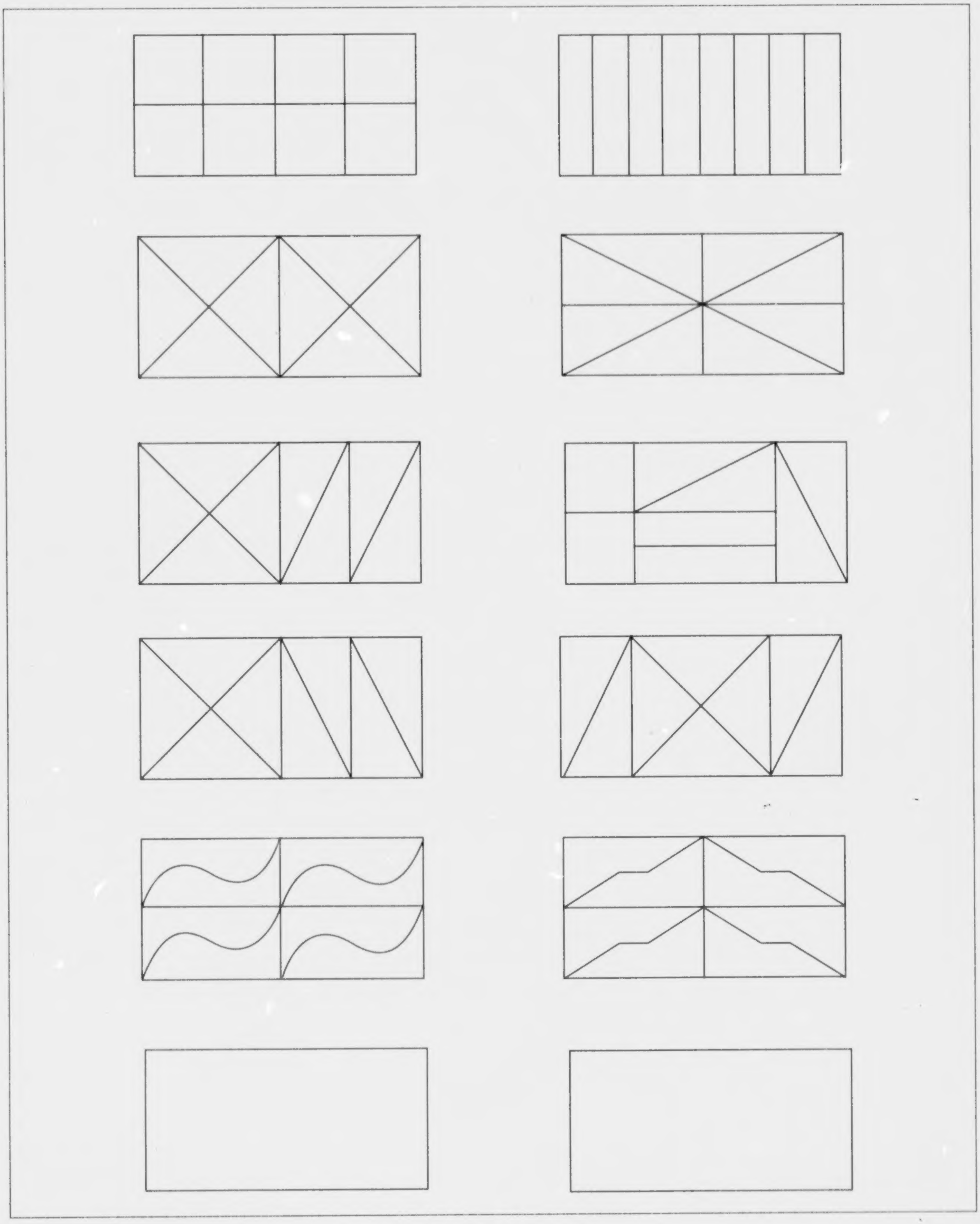
***Concluding Extension: Project and Writing Prompts***

Have students plan a field trip to a local business to help them make connections between what they have learned in the classroom about rational numbers (including fractions, decimals, and percents) and how rational numbers are used in the real world. Or, they could invite a member of the local business community to discuss with the class how rational numbers are used in his or her business. For example, as part of a visit to a grocery story, students could talk to employees in the meat, produce, and deli departments to discuss how these items are weighed (usually on scales with decimal displays) and how customers often estimate and compare the items (usually by using fractional amounts). For example, depending on the location of the store, customers may ask for half a kilogram of turkey or half a pound of turkey, but the scale may read 0.58 kilograms or 0.58 pounds. Students visiting a retail store could talk with employees about prices, sale prices based on certain discounts expressed in percents, and calculating tax.

As a summary of what they have learned, students could prepare a written report that explains the connections they see between what they have learned about rational numbers in school and how rational numbers are used in the businesses they visited or by the business representative who visited the class. Teachers can use students' reports to assess students' understanding of rational numbers and of the usefulness of rational numbers in the real world.

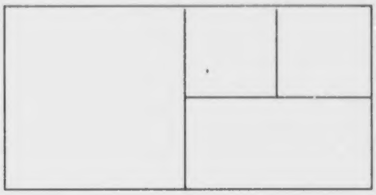
BEST COPY AVAILABLE

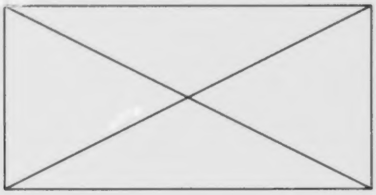
# BENNY'S CAKES—WAYS TO MAKE EIGHTHS

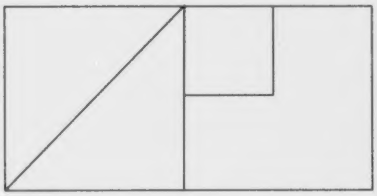


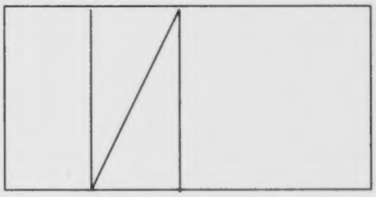
**OVERHEAD MASTER 2.1**

## BENNY'S CAKES—WAYS TO MAKE FOUR PIECES





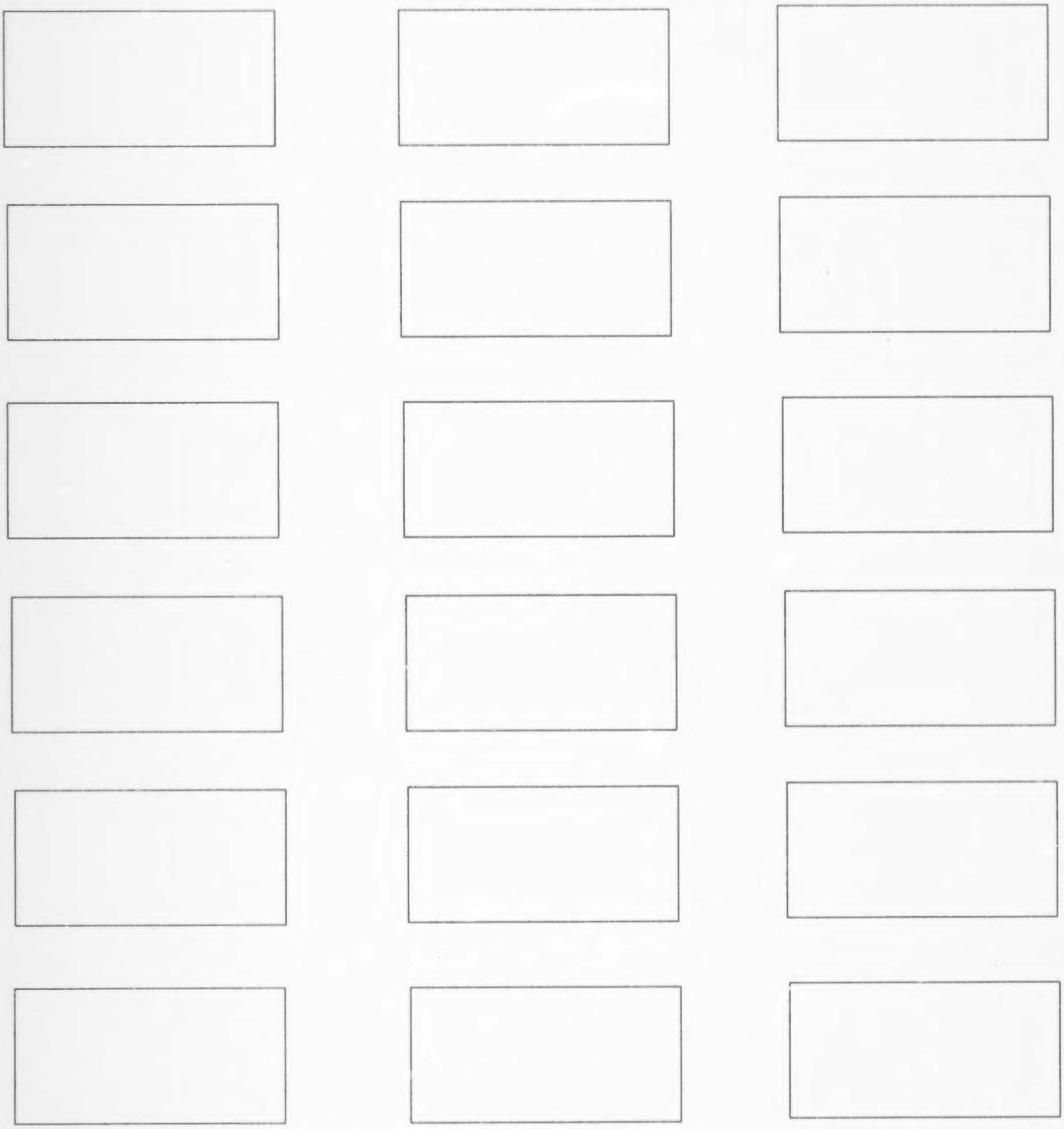




**OVERHEAD MASTER 2.2**

Name ______________________ Date ______________

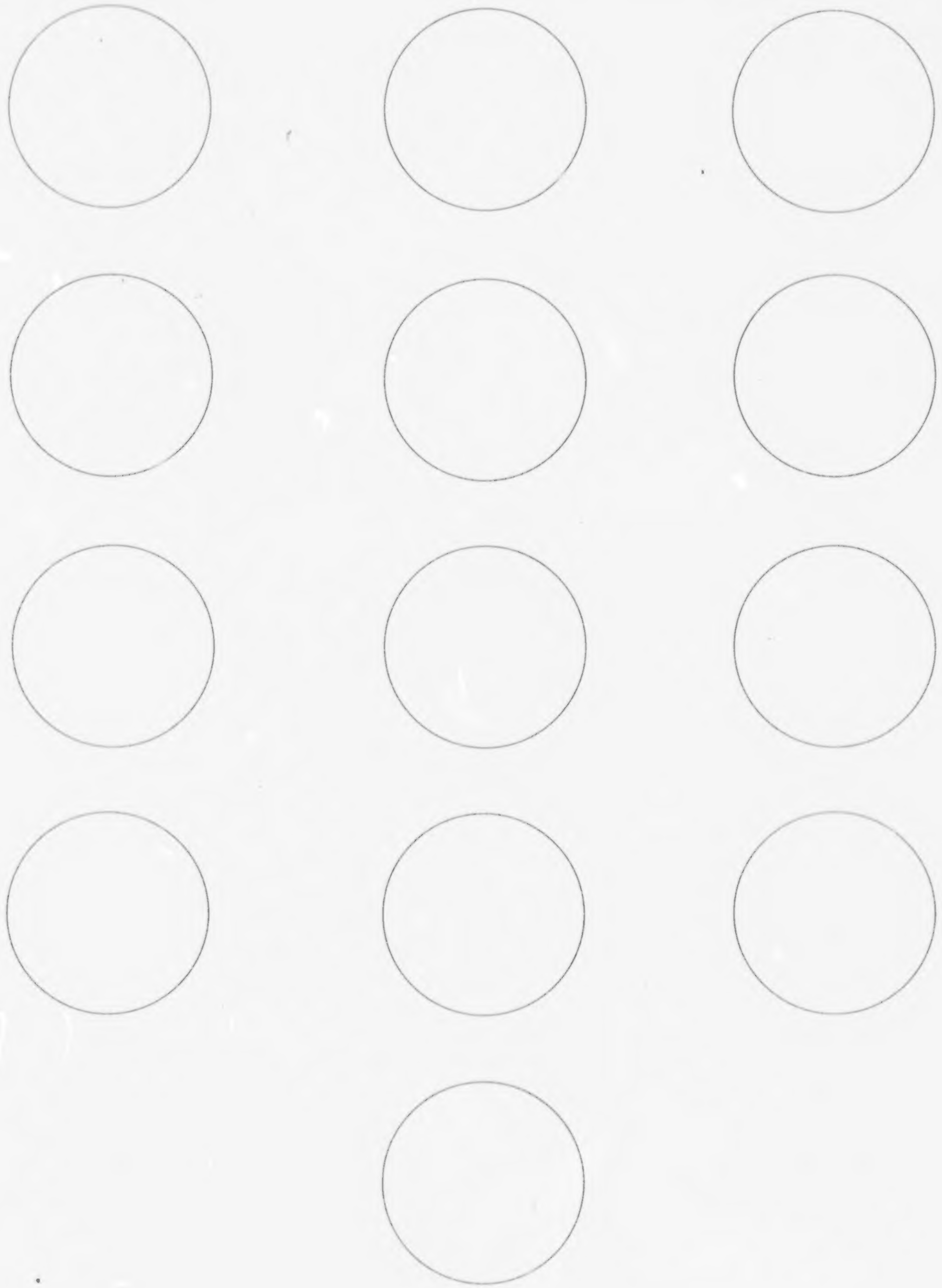
## BENNY'S CAKES



**ACTIVITY SHEET 2.1**

Name ______________________________ Date ______________

**SHARING A BAKER'S DOZEN COOKIES WITH YOUR FAMILY**



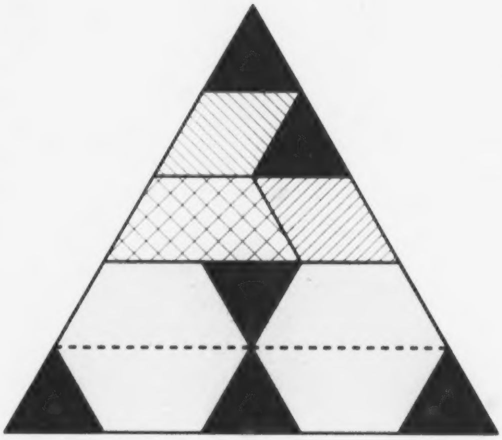
13 cookies shared by ____ people results in ____ cookies for each person.

**ACTIVITY SHEET 2.2**

Name ______________________ Date ____________

**PATTERN BLOCK CAKE 1**

If each triangular piece costs $1.00, how much will the whole Pattern Block Cake cost?



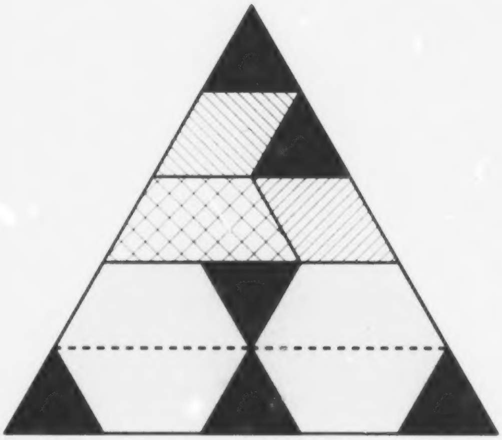
| Name of Piece | Cost |
|---|---|

**ACTIVITY SHEET 2.3**

Name ______________________________ Date ______________

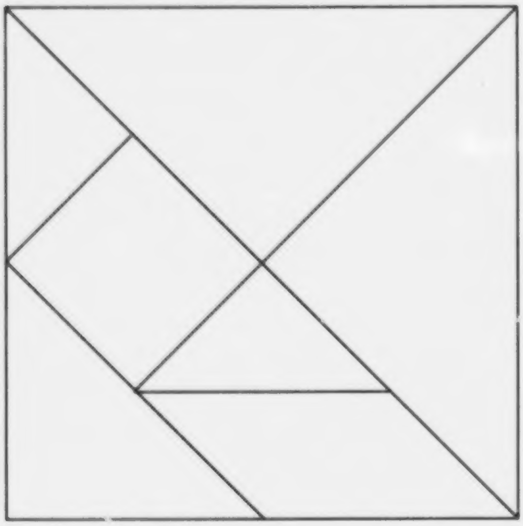
## PATTERN BLOCK CAKE 2

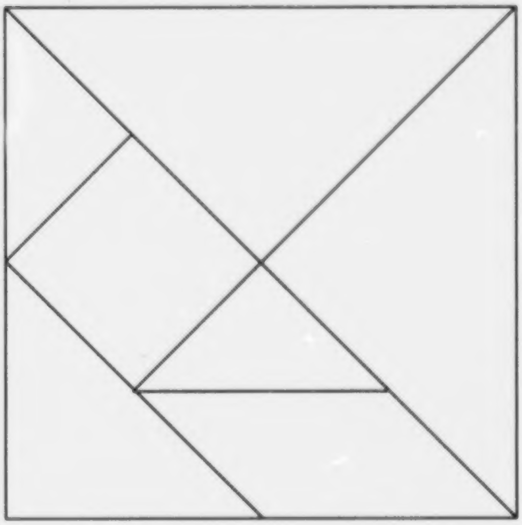
If each whole Pattern Block Cake costs $1.00, how much will each piece cost?

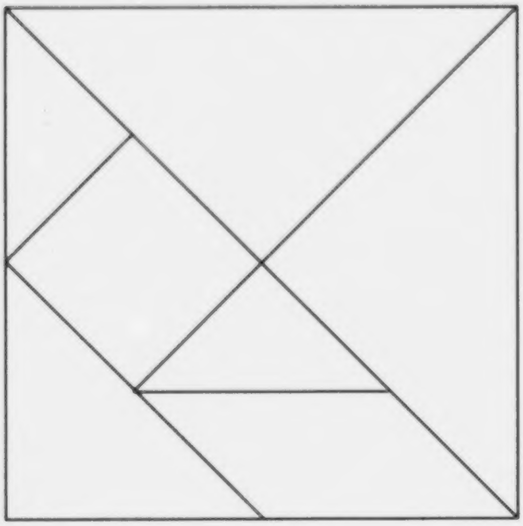


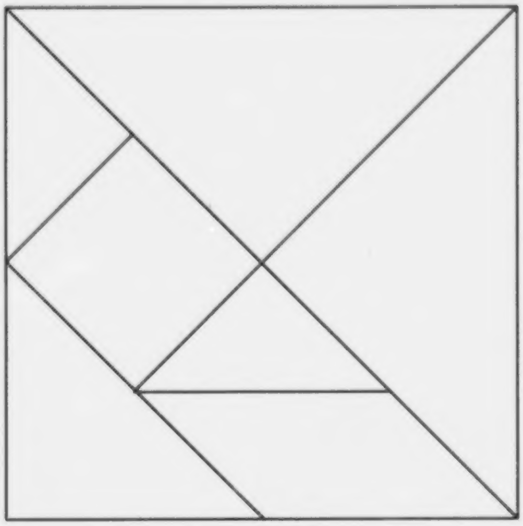
| Name of Piece | Fraction of Cake | Cost |
|---|---|---|

## TANGRAM CAKES







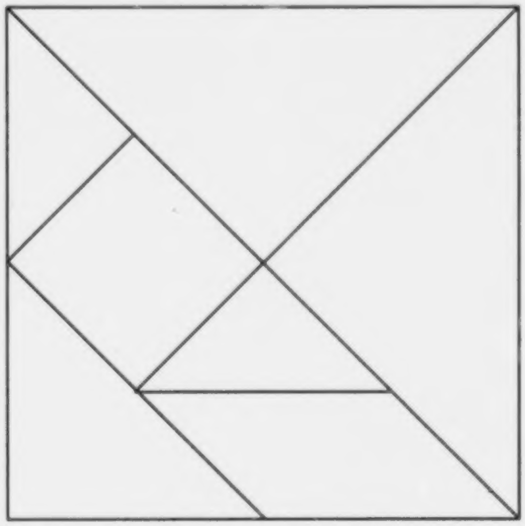


**OVERHEAD MASTER 2.3**

Name ____________________ Date ____________

**TANGRAM CAKES**

If this Tangram Cake costs $1.00, how much will each piece cost?



| Name of Piece | Fraction of Cake | Cost |
|---|---|---|

**ACTIVITY SHEET 2.5**

## CLUSTER B: APPLYING RATIONAL NUMBER AND PROPORTION CONCEPTS

This cluster contains three investigations: Garbage and Recycling: The Rational (Number) Approach; The M&M Experiment: A Sweet Approach to Ratio and Proportion; and Sharks in a Box: A Tagging-and-Recapturing Simulation. These three investigations develop and expand students' notions of ratio and proportion in the context of data collection and data analysis activities.

Garbage recycling is a real-world issue that affects everyone. The first investigation is presented as a five-part project in which students analyze recyclable and nonrecyclable portions of garbage generated by themselves and their families. The students become involved in formulating a problem related to recycling trash and collecting and analyzing data about the kinds and quantities of trash they generate. On the basis of the data they collect from their classmates, students extrapolate information about the amount of garbage generated by people in their community, their city, and their state or province. The situation presented provides a natural setting to make connections among environmental science, data analysis, and rational numbers. Ratios and proportions are used to compare recyclable, nonrecyclable, and total garbage. Equivalent fractions, decimals, and percents are presented throughout the project tasks.

The second investigation contains three activities in which data about M&M candies are analyzed. Ratios, equivalent fractions, and proportions are the mathematics concepts reviewed, developed, and expanded as students eventually make predictions from the data they collect. Depending on students' experiences with ratio and proportion, teachers may want to present Investigation 4: The M&M Experiment before having them work on Investigation 5: Sharks in a Box, since the former activity is designed to prepare students for the simulation. In Investigation 5, students' knowledge of ratio and proportion is applied in solving equations to predict the total of a population on the basis of information from a sample of the population. This simulation of a real-world situation employs relative frequency to estimate chance. For other related data collection and analysis activities, see *Dealing with Data and Chance* (Zawojewski 1991).

### INVESTIGATION 3: GARBAGE AND RECYCLING: THE RATIONAL (NUMBER) APPROACH

*Francine Sicklick and Susan B. Turkel*

To help students see the role of mathematics in understanding a real-world problem and see mathematics as an interdisciplinary tool, the activities in this project guide students' grasp of the enormity of the problem of garbage production and disposal in the modern world. The focus of the activities is on understanding a crucial problem from environmental science and on applying the mathematics of ratios and percents to understand the challenges to environmentalists.

The project, titled You and Your Trash, contains five parts: formulating the problem, collecting data, compiling data, analyzing data, and extrapolating from data. Students will measure garbage production in their own homes and, using mathematics, derive from these results data for personal, class, town, and state or province garbage production over the course of a day, a week, and a year.

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

The project may be assigned to individual students, to small groups of three or four students, or to the whole class. The five parts may be presented individually in sequence so that the results of each part can be reviewed, or, depending on students' ability, the five-part project may be given to students in its entirety and used as a project assessment.

> ***"A primary goal in choosing curricula and teaching methods in the middle grades should be the disciplining of young adolescents' minds, that is, their capacity for active, engaged thinking.... A disciplined mind connotes a disposition toward inquiry, discovery, and reasoning" (Carnegie Council 1989, p. 43).***

*Prior knowledge:* Metric or English units of measure for weight; awareness of percent; ability to compute the mean of a set of data; ability to create and interpret a circle graph; ability to use a calculator

*Concepts:* Estimation, ratio, proportion, decimals, percents, circle graphs, data collection and interpretation

*Objectives:*

1. To formulate a problem related to garbage production and recycling
2. To collect and organize data
3. To compare estimates with actual data
4. To use ratios to compare recyclable and nonrecyclable garbage to total garbage
5. To use equivalent fractions to develop an understanding of proportions
6. To use percents to analyze garbage production
7. To compile, interpret, and analyze data
8. To extrapolate from data

### Part 1: Formulating the Problem

*Materials:* Newspapers with articles on garbage and recycling; Activity Sheet 3.1 (glossary); Activity Sheet 3.2 ("The All-Consuming Lifestyle" by Irene Virag) for each student; Activity Sheet 3.3

How did we, as individuals, as a nation, and as a world, reach this crisis moment with respect to garbage and recycling? What are the rational and safe solutions? Through conservation, by wasting fewer resources, by using fewer packaging materials, and so on, solid waste can be reduced. Undoubtedly, when materials are recycled, less waste is disposed of, and, as a bonus, natural resources will be saved.

Hardly a week goes by that solid waste and depletion of resources are not in the news. However, the headlines that glare at our students may not be completely understood by them. Each article is filled with numbers—usually ratios and percents—measuring the progression of the problem and the extent of its remediation. When goals are set for recycling programs, they are often presented as percentile reductions in a given time frame. The problem is impossible to grasp without an understanding of the meaning of these numbers and the relationships implied.

This is part 1 of a five-part project. Working individually or in small groups, students become aware of the magnitude of the problem and formulate the problem in their own words. To give students a perspective on the problem of waste generation and its disposal, they read articles they collect from newspapers, compare headlines, and review some "important facts to consider" on Activity Sheet 3.3. The references for these facts can be found in the section on recycling in the selected bibliography ([a] Indiana State Education Department 1991; [b] *Litter Control* 1985; [c] *Rush to Burn* 1989; [d] Schwartz 1989; [e] Rybczynski 1992). Have copies of the glossary available if students are

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

unfamiliar with the vocabulary. All of this should lead to the statement of the large problem on Activity Sheet 3.4 and motivate students to start thinking about what they need to do to find out how garbage reduction will affect them and their lifestyles.

*Diagnostic Assessment:* By reading through students' written descriptions, the teacher can determine students' familiarity and understanding of such mathematical relationships as ratios and percents. Also, students' interpretation of the problem can be reviewed and discussed.

### Part 2: Collecting Data

*Materials:* Activity Sheet 3.4; five grocery store plastic bags for each student each day; calculators; a bathroom scale or some instrument to weigh garbage at home

The problem that drives the next four parts of this project is stated at the top of Activity Sheet 3.4: Suppose your state or province wants to reduce its garbage production by 10 percent by the turn of the century and recycle 40 percent. How will this affect your lifestyle?

Students describe what "10 percent" and "40 percent" mean. This activity can be used as a diagnostic assessment to determine the level of students' understanding of percent. To personalize the problem and provide a motivation for students to "invest" in the problem, students estimate their own and their family's garbage production.

Students may need some advice about how to weigh the garbage produced for the seven days. Students first weigh the total garbage each day (including the nonrecyclables). They then separate the garbage and weigh each bag in the following manner. Each day, students use the five grocery store plastic bags to separate the garbage: one bag for paper, one for aluminum, one for glass, one for plastic, and one for all other garbage, which is considered the nonrecyclable garbage. (Students might need to borrow scales from school if they do not have access to a bathroom scale.) The bags may be a bit unwieldy for a bathroom scale; students can first weigh themselves and then weigh themselves holding the bag of garbage. The difference between the two measurements equals the weight of the bag of garbage.

Have students complete the prediction column before they complete other parts of the table. It is revealing for them to compare their actual data with their estimates and predictions.

In item 4, determining the quantity of garbage generated by an individual, expect students to divide the total weight of their family's garbage by the number of people in their family. This procedure assumes that everyone in the family produces the same amount of garbage. Ask them under what conditions the assumptions are valid or invalid. Some factors that influence garbage production include house guests, party guests, sickness, diet, and vacations. A writing prompt is included to give insight into students' thinking about the measuring process.

### Part 3: Compiling Data

*Materials:* Activity Sheet 3.5; calculators (one for each student)

Students will need the data they collected in part 2. If the entire class or a small group has been working on this project, this is an opportunity for them to compile their data.

Encourage students to use calculators to compute class totals and means once the family amounts have been entered.

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

### *Part 4: Analyzing Data*

*Materials:* Activity Sheet 3.6; compasses; angle ruler or protractor; calculator (one for each student)

Students record the means they computed in part 3 in the table on Activity Sheet 3.6.

A *ratio* results when we compare two quantities multiplicatively. The quantities are of like nature (e.g., kilograms compared with kilograms). A ratio is a numerical expression of how much there is of one quantity in relation to another quantity (Thompson 1994). A ratio can also be defined as an ordered pair of measurements (Hoffer and Hoffer 1992). A ratio can be expressed as 5 kilograms of paper:20 kilograms of garbage; written as a fraction, 5 kg/20 kg; and simplified to 1/4. The same ratio can be expressed as a decimal, 0.25. A percent is a ratio expressed as parts of one hundred, or per hundred. When written as a percent, the example, 5 kg/20 kg, is 25 percent. To help students relate percents to fractions, highlight "benchmark" fractions. For example, 19.5 percent is close to 20 percent, which equals 1/5; 25.4 percent is close to 25 percent, which equals 1/4; 11.9 percent is close to 10 percent, which equals 1/10.

*Note:* Because of rounding errors, the sums of the fractions or the decimals may not equal 1. Also, the sum of the percents may not equal 100 percent. Discuss this situation with students.

Sometimes students confuse ratio and rate. A *rate* is a ratio between two unlike quantities (e.g., miles per hour; words per minute).

The students illustrate the data collected in part 4 by using a circle graph. To construct a circle graph, students must know that a circle has 360 degrees. The portions of the circle will be determined on the basis of the ratios that express the relationship between the weight of each of the five garbage types and the total weight of the garbage (i.e., total weight of paper:total weight of garbage; total weight of aluminum:total weight of garbage; total weight of glass:total weight of garbage; total weight of plastic:total weight of garbage; total weight of all other garbage:total weight of garbage). If the ratio of the total weight of paper to the total weight of garbage is 1/4, the students multiply 360 degrees by 1/4 to obtain the number of degrees (90) in the portion of the circle that represents this ratio. Students need angle rulers or protractors to construct the portions of the circle.

Students may also use equivalent fractions to find the number of degrees in the portion of the circle that represents this ratio. For example, 90 degrees satisfies the open sentence, 1/4 = ?/360. For further discussion about constructing circle graphs, see Curcio (1989). See figure 3.1 for a sample of a seventh grader's circle graph.

*Comments about the student's work:* Clearly, this student is starting to understand the concept of percent as a part of a whole; however, the concept as related to the circle graph is not completely understood and needs further clarification in determining the correct size of each circle sector. For example, the student neglected to provide a percent for the "other garbage" sector and does not seem to understand that the sum of the sectors should not be greater than 100 percent. The teacher needs to address this idea with the student.

In item 8 on Activity Sheet 3.6, students are asked to compare their circle graphs with the graph of garbage produced by the Sigmann family. Before comparing the data, it is essential for students to generate the

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

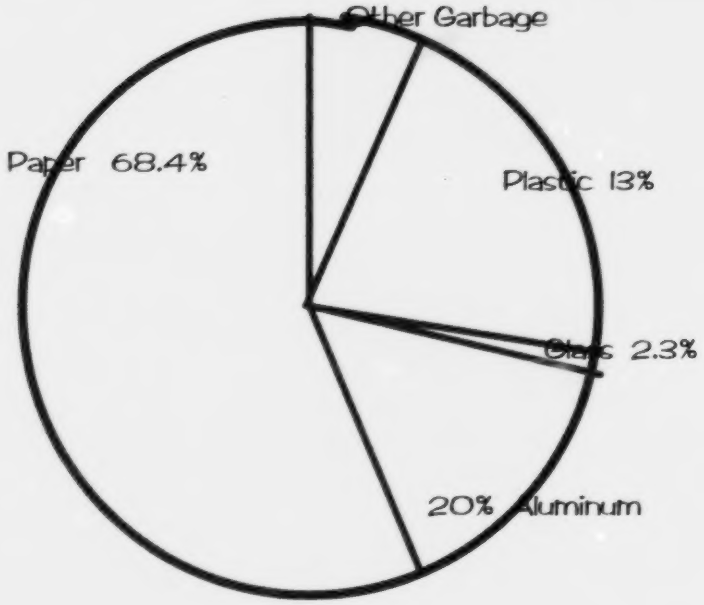


*Fig. 3.1*

assumptions that will be necessary when they generalize from the class's situation to their town, state, or province. Students must recognize that the data were collected during the course of four weeks. They should also categorize some garbage types, such as newspapers and paper, before making comparisons. See figure 3.2 for a sample of a seventh grader's comparison. The teacher's commentary is below.

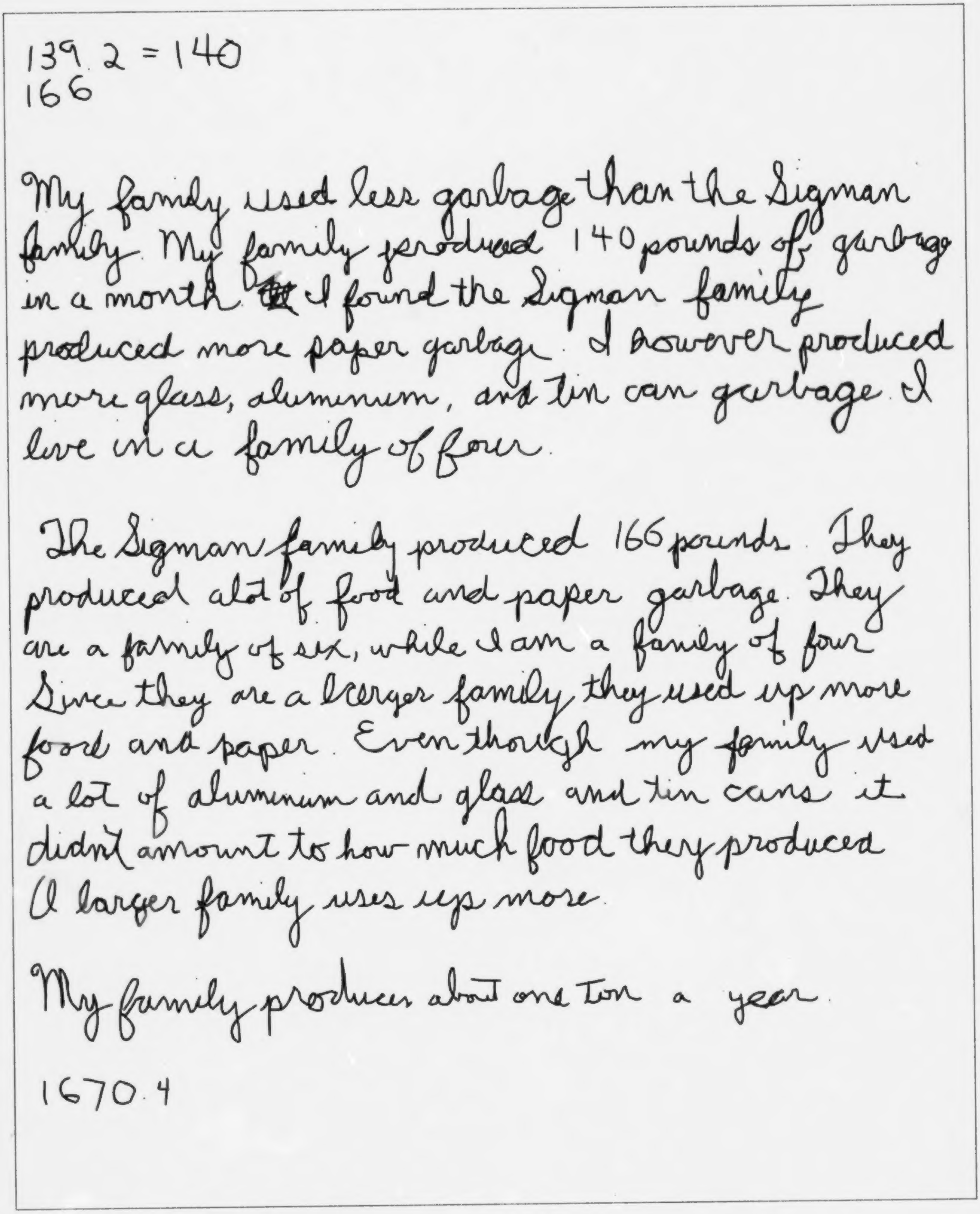
*Comments about the student's work:* This student was able to interpret, analyze, and compare her family's garbage facts with those of the Sigmann family. The student expanded her interpretation and drew valid conclusions to explain the differences. Writing gives students the opportunity to clarify and concretize their thoughts.

### Part 5: Extrapolating from Data

*Materials:* Activity Sheet 3.7; calculators (one for each student)

Students are asked to find out how much garbage is generated by their town and state or province. To obtain this information students will need to find the population of their town and state or province. This information can be obtained from census figures, almanacs, or a local newspaper publisher. In addition, students could contact local government agencies.

A *proportion* is the equality of two ratios. Proportions can be used to find equivalent percents as well as the number of degrees in the portion of a circle that represents a particular ratio, as in the example in part 4. For example, expressing the ratio 5 kg/20 kg as a percent can be done by setting up a proportion, 5:20::?:100, which can also be expressed as 5/20 = ?/100. Equivalent fractions, the unit rate method, or a calculator could be used to find the missing element. An example of the equivalent fraction method and the unit rate method can be found in the introduction on page 4.



139.2 = 140
166

My family used less garbage than the Sigman family. My family produced 140 pounds of garbage in a month. I found the Sigman family produced more paper garbage. I however produced more glass, aluminum, and tin can garbage. I live in a family of four.

The Sigman family produced 166 pounds. They produced alot of food and paper garbage. They are a family of six, while I am a family of four. Since they are a larger family, they used up more food and paper. Even though my family used a lot of aluminum and glass and tin cans it didnt amount to how much food they produced. A larger family uses up more.

My family produces about one ton a year.

1670.4

**Fig. 3.2. Example of a seventh grader's comparison**

Students should use proportions to extrapolate the weight of the garbage for the class's families to that of the weight of the garbage in their town and in their state or province. For example, if the total weight of garbage produced by twenty-four families of four in one week was 320 kg, the amount of garbage produced by a town with 400 000 people can be extrapolated by using the following proportion:

$$\frac{96}{320} = \frac{400\,000}{n}, \quad n = 1\,333\,333.3 \text{ kg in one week!}$$

Proportions can be set up to determine the weight of garbage produced in one week by the people living in the state or province.

### *Extensions*

1. What percentage of town garbage do students personally contribute each year? What percentage of state or province garbage does your town contribute?

2. Students might want to find out how much garbage their school generates in a day, a week, and a school year. How much of the school's waste is recyclable? Is it reasonable to suggest that the school recycle 40 percent of its garbage?

3. Larry Berger, an animal nutrition expert from the University of Illinois, has found that newspapers made with soybean oil–based ink, which is not harmful if ingested, contain cellulose. Cows, he suggests, could eat boiled newspapers and digest the energy-rich molecules from the cellulose. If a cow eats ten pounds of boiled newspapers a day, 40 percent of its diet could be replaced! There are approximately 30 million cows in the United States. If each cow had a diet that was 20 percent newspapers, could we use up all our unwanted newspapers? ("TNT News Blasts," July/August 1992, p. 2)

4. "In the last 50 years, the amount of waste generated by each person in the United States has doubled. The United States Environmental Protection Agency (EPA) expects America's waste generation to increase 20% in the next ten years." (*Waste Away Information Activities for Investigating Trash Problems and Solutions for Upper Elementary and Junior High School Students*, 1989, p. 2) Approximately how much waste will each person generate ten years from now?

Discuss with students what their future data (ten years from now) might look like. Allow students to arrive at mathematical results and discuss influences that might realistically alter these figures, such as population trends, demographics, lifestyle changes, and so on.

5. Students might want to consider the following question: How much of a difference in your overall estimates of yearly and community consumption would be expected from a 1 percent, 10 percent, or 100 percent error in your initial weights?

*Performance Assessment:* As a culminating activity, you may want to supply students with a table of data similar to the type they collected and ask them to construct a circle graph showing the various types of garbage, to label each portion of the graph with the appropriate percent, to formulate an assertion that could be supported by these data, and to explain the graph in writing. This performance assessment might be done either individually or in small cooperative groups. Give students calculators to expedite computation.

## GLOSSARY

| | |
|---|---|
| *Biodegradable* | can be reduced to harmless products by organisms |
| *Decompose* | to break up or separate into basic parts |
| *Disintegrate* | to separate into parts or fragments |
| *Environmentalist* | someone concerned about the human environment |
| *Pollution* | impurities and contamination by human-made waste |
| *Recycle* | to pass through a cycle or to use again and again |
| *Solid Waste* | discarded material not in a liquid or a gaseous form |

**ACTIVITY SHEET 3.1**

## THE ALL-CONSUMING LIFESTYLE

*Irene Virag*

It happened, as it always does, without anyone in the Sigmann family even noticing.

At six in the morning, Deneen Sigmann was half asleep in her four-bedroom ranch on Long Island when she opened her first pack of Salem Lights and threw the cellophane wrapper into the trash. It floated as silently as an autumn leaf into the 30-gallon plastic bag that lines the brown Rubbermaid garbage can in the kitchen. The wrapper joined the previous day's newspaper, a used Brillo pad, a mound of left-over ziti, an empty shampoo bottle, a crumpled bag of potato chips, a broken glass, a banana peel, a plastic yogurt container, four packets of Sweet 'N Low, and a half-full can of beef stew for dogs.

Deneen Sigmann lit a cigarette and headed into the bathroom. As she left the kitchen, her children were fixing their own breakfasts. Twelve-year-old Jennifer poured the last of the Cheerios into a bowl and threw the box away. Sixteen-year-old Tom tore open two individually wrapped store-bought fudge brownies. He ate the cakes, wiped his mouth with a paper towel, and tossed a wad of paper packaging into the Hefty Cinch-Sak. Eight-year-old Cathy finished the Count Chocula cereal and the milk—then she shoved the cardboard box and one-gallon plastic jug into the bulging garbage bag.

When Deneen Sigmann returned to the kitchen, the garbage can was close to overflowing. Waltie, Deneen's 21-year-old son, made a brown-bag lunch—four slices of stale white bread and the plastic wrap from the bologna were added to the can. Deneen drank her fourth cup of black coffee, then dumped the grounds. She emptied an ashtray—a cigarette butt stuck in a glob of spaghetti sauce that had been discarded the night before, and the ashes from two packs of Salems descended like a fine snow that hides the litter underneath.

Before leaving for work, Walter Sigmann, Deneen's husband, stuck his hand into the bag and shoved the contents deeper into the Rubbermaid can. He crushed cereal boxes and milk containers. He squished the pasta into the cherry-vanilla yogurt. He made room for more garbage.

Later that weekday evening, sometime between the after-dinner cleanup and bedtime, Walter Sigmann pulled the 30-gallon Hefty Cinch-Sak out of the Rubbermaid trash can. He carried it outside and plopped it down in front of the garage next to three equally stuffed bags, one of which had been ripped open by a hungry dog or maybe a raccoon. An onion and an apple core lay on the driveway.

By Sunday, six bags blocked the garage door. Walter Sigmann piled the week's worth of garbage into his van. He drove five miles to the Brookhaven Town landfill and waited in line 45 minutes to pay a $2 entry fee. He added his bags to the vast wasteland.

When Walter Sigmann returned home, a clean 30-gallon plastic bag lined the trash can in the kitchen.

A *Newsday* article, reprinted by permission. Newsday, Inc., Copyright 1989

**ACTIVITY SHEET 3.2**

Name ______________________ Date ____________

***YOU AND YOUR TRASH***

## Part 1: Formulating the Problem

1. Read through some newspapers and collect articles about garbage and recycling. Compare your headlines with these:

**LEADERS OF THE WORLD CONVENE ON AN EARTH SUMMIT**

**BARGES OF GARBAGE ARE UNABLE TO FIND A LANDING SPOT**

**LANDFILLS ARE CLOSING AND ALTERNATIVE APPROACHES FOR SOLID WASTE REMOVAL ARE EXPLORED**

2. In your own words, write a description of the problem the world has with garbage and recycling. Include numerical facts and any mathematical relationships, such as ratios and percents, that are presented. Describe what all of this means to you.

3. Read "The All-Consuming Lifestyle," by Irene Virag, and some "Important Facts to Consider," below. Identify any vocabulary words that are new for you.

### Important Facts to Consider

(a) Plastic loop six-pack can holders can take up to 450 years to decompose.

(b) Flip-top tabs from aluminum cans can take from 100 to 500 years to disintegrate.

(c) Anything that is not biodegradable will be with us for centuries.

(d) Fourteen million pounds of paper are used to produce the *New York Times* and the *New York Daily News* on an average Sunday. It takes about seventeen trees to produce one ton of paper.

(e) Papers take up over 40 percent of the contents of landfills by volume.

4. Write some solutions to the problem you described in 2, above.

***ACTIVITY SHEET 3.3***

Name ______________ ______________ Date ______________

## YOU AND YOUR TRASH

### Part 2: Collecting Data

Suppose your state or province wants to reduce its garbage production by 10 percent by the turn of the century and recycle 40 percent. How will this affect your lifestyle?

1. What does a "10 percent reduction" mean to you? Give an example. What does "recycle 40 percent" mean to you? Give an example.

To understand your state or province goals, you must know the amount of garbage you and your family produce.

2. Estimate the amount of trash you and your family generate. How many pounds or kilograms of garbage do you think you produce each day? Each week? Each year? Write your estimates below.

I produce about ______________ of garbage each day.

I produce about ______________ of garbage each week.

I produce about ______________ of garbage each year.

How many pounds or kilograms of garbage do you think your family produces each day? Each week? Each year? Write your estimates below.

My family produces about ________ of garbage each day.

My family produces about ________ of garbage each week.

My family produces about ________ of garbage each year.

3. Each day for one week collect information about the garbage your family produces. To weigh the garbage each day, you will need five grocery store plastic bags. Use one bag for paper garbage, one bag for aluminum garbage, one bag for glass garbage, one bag for plastic garbage, and one bag for all other garbage. First, find the *total* weight of *all* the garbage produced. Perhaps you can use your bathroom scale. If you find weighing the garbage on your bathroom scale difficult, you can weigh yourself first and then weigh yourself holding all the bags of garbage. How would this help you determine the weight of the garbage? Repeat this step for each individual bag containing paper, aluminum, glass, or plastic garbage. Determine the weight of the remaining bag.

**ACTIVITY SHEET 3.4**

Name ______________________ Date ______________

**YOU AND YOUR TRASH**

## Part 2: Collecting Data

Use the table below to record the data you collect during the seven days. Each day, fill in your prediction *before* weighing the garbage.

Weight of Garbage Generated by My Family

| Day | Prediction | Total Weight of Garbage | Weight of Paper Garbage | Weight of Aluminum Garbage | Weight of Glass Garbage | Weight of Plastic Garbage | Weight of All Other Garbage |
|---|---|---|---|---|---|---|---|
| 1 | | | | | | | |
| 2 | | | | | | | |
| 3 | | | | | | | |
| 4 | | | | | | | |
| 5 | | | | | | | |
| 6 | | | | | | | |
| 7 | | | | | | | |
| Totals | | | | | | | |

During the week, answer the following questions:

1. Explain any difficulties you encounter as you determine the amount of garbage in your household by weighing it. For example, what assumptions, adjustments, or estimates had to be made?

2. What would happen if the amount of paper garbage for a given day was so little that you seemed to weigh the same on the bathroom scale with and without the bag of paper? Would you record "0"? What could you suggest to get around this problem of underestimating?

3. How reliable is your bathroom scale? What if you stepped on it three different times and got slightly different answers each time? What could you do to improve the accuracy of your data?

**ACTIVITY SHEET 3.4 (CONTINUED)**

Name ______________________________ Date ______________

***YOU AND YOUR TRASH***

## Part 2: Collecting Data

4. Using the data in your table, determine approximately how much garbage you personally generate in a day. In a week? In a year? Describe how you would determine these amounts from the table. What assumptions are you making by determining these amounts this way? Under what conditions are the assumptions valid or invalid?

I produce approximately ________ of garbage in one day.

I produce approximately ________ of garbage in one week.

I produce approximately ________ of garbage in one year.

How do these estimates compare with your predictions and your estimates in question 2?

How do the data about your family's garbage that you collected in one week compare with your estimate in question 2?

5. In your math journal, select one of the seven days and describe the activities that account for the contents of the garbage. How do the amounts of garbage in the separate categories reflect your lifestyle? What changes in your lifestyle would affect changes in this garbage production? What changes can you make that would lead to less solid waste going to the garbage heap? What changes do you think industries could make to help you? How could you get them to make these changes?

***ACTIVITY SHEET 3.4 (CONTINUED)***

Name ______________________________ Date ______________

**YOU AND YOUR TRASH**

## Part 3: Compiling Data

You will need the data you collected in part 2 of this project. Working with a small group of your classmates or with the data collected by the entire class, find the mean (the arithmetic average) amount of garbage produced by the families of all the students in the class.

Total Weight of Garbage Generated by All Families in One Week

| Student | Total Weight of Garbage | Weight of Paper Garbage | Weight of Aluminum Garbage | Weight of Glass Garbage | Weight of Plastic Garbage | Weight of All Other Garbage |
|---|---|---|---|---|---|---|
| | | | | | | |
| | | | | | | |
| | | | | | | |
| | | | | | | |
| | | | | | | |
| | | | | | | |
| | | | | | | |
| | | | | | | |
| | | | | | | |
| | | | | | | |
| | | | | | | |
| | | | | | | |
| Total | | | | | | |
| Mean | | | | | | |

**ACTIVITY SHEET 3.5**

Name ______________________________ Date ______________

## *YOU AND YOUR TRASH*

### Part 4: Analyzing Data

Insert the means computed in part 3 in the table below.

Ratios of Mean Weights of Garbage Types to
Mean Weight of Total Garbage Generated by All Families

| | Weight of Paper Garbage | Weight of Aluminum Garbage | Weight of Glass Garbage | Weight of Plastic Garbage | Weight of All Other Garbage |
|---|---|---|---|---|---|
| Mean | | | | | |
| Ratio of each weight to the mean weight of total garbage expressed as | | | | | |
| a fraction | | | | | |
| a decimal | | | | | |
| a percent | | | | | |

1. a) What is a ratio?

b) Give an example of a ratio.

c) Express the ratio of the mean weights of the types of garbage to the mean weight of the total garbage generated by the class. In the table above, enter these ratios expressed as a fraction, and then use your knowledge of decimal equivalents or a calculator to convert them to decimals.

2. a) What is a percent?

b) Give an example of a percent.

c) Convert the ratios entered in the table above to percents. Describe how you determined the percents.

3. When you add the ratios of each mean weight expressed as a fraction, what is the sum? ____ When you add the ratios of each mean weight expressed as a decimal, what is the sum? ____ When you add the ratios of each mean weight expressed as a percent, what is the sum? ____ Share your results with your classmates. What do you observe?

4. What percent of the total weight of garbage generated by the class is recyclable?

5. What percent of the total weight of garbage generated by all the families is nonrecyclable?

**ACTIVITY SHEET 3.6**

Name ______________________ Date ____________

***YOU AND YOUR TRASH***

**Part 4: Analyzing Data**

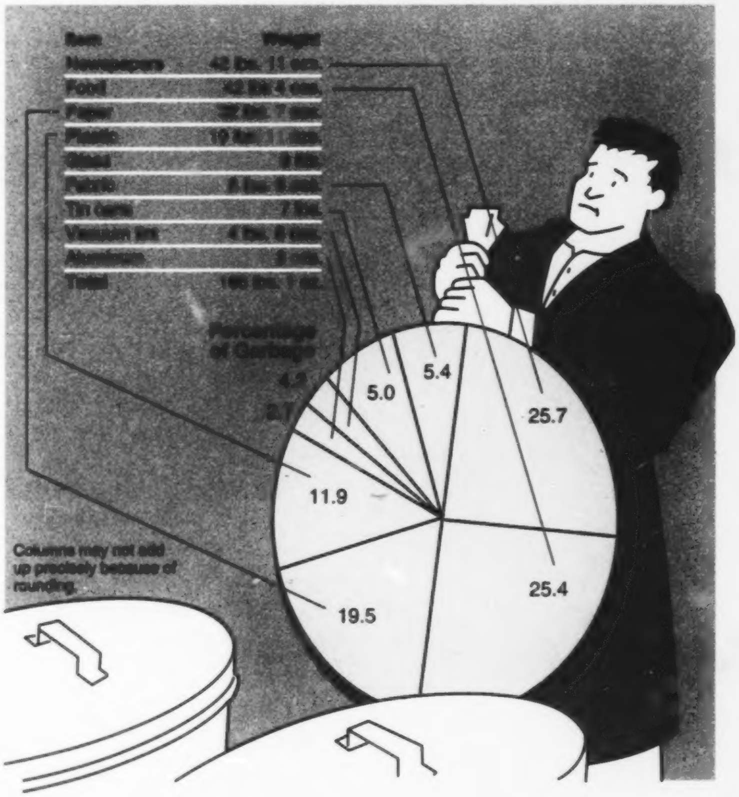
6. Use a compass to construct two circles.
   a) Using one of the circles and the data you collected in part 2, construct a circle graph of the amount of garbage your family produced in one week. First, convert the data to percents. Second, using an angle ruler or a protractor, divide the circle into portions to represent the percent for each type of garbage. Finally, label the portions of the circle and create a title for the graph.
   b) Using the second circle and the data from part 3, construct a circle graph of the mean amount of garbage your class produced in one week. The data have already been converted to percents. Using an angle ruler or a protractor, divide the circle into portions to represent the percent for each type of garbage. Label the portions of the circle and create a title for the graph.

7. Using the circle graphs, describe the differences and similarities between the amount of garbage your family produces and the means of the totals for the class results.

8. Compare the circle graphs you made with the circle graph of the garbage produced by the Sigmann family. For example, how do the types of recyclable garbage compare? How does the amount of nonrecyclable garbage compare?

## One Family's Trash

Over the course of four weeks, Newsday collected and analyzed 12 bags of garbage from the six-member Sigmann family of Holbrook. The garbage added up to 166 pounds, the equivalent of an average-sized man. Here is what the Sigmanns threw out:



A *Newsday* graph, reprinted by permission. Newsday, Inc., copyright 1989

Name ______________________________ Date ______________

***YOU AND YOUR TRASH***

## Part 5: Extrapolating from Data

1. Find the population of the town in which you live: __________

2. Find the population of the state or province in which you live: __________

3. a) What is a proportion?

   b) Give an example of a proportion.

4. Describe how you can estimate the amount of garbage produced in one day, one week, and one year by the people in your town and the people in your state or province. Use this method to complete the table below.

| | Weight of Garbage Generated | | |
|---|---|---|---|
| | My Class's Families | My Town | My State or Province |
| Population | ________ | ________ | ________ |
| Weight of garbage | | | |
| in 1 day | ________ | ________ | ________ |
| in 1 week | ________ | ________ | ________ |
| in 1 year | ________ | ________ | ________ |

Let's return to the original problem: "Suppose your state or province wants to reduce its garbage production by 10 percent by the turn of the century and recycle 40 percent."

5. Is it reasonable to expect recycling of 40 percent of garbage by the turn of the century? How do you think this can be done?

6. How much of your garbage today can be recycled?

7. By weight, what is 10 percent of your garbage? What methods can you suggest for a 10 percent reduction of solid waste? How would this affect your lifestyle?

***ACTIVITY SHEET 3.7***

## *INVESTIGATION 4: THE M&M® EXPERIMENT: A SWEET APPROACH TO RATIO AND PROPORTION*

*Frances R. Curcio and J. Lewis McNeece*

By taking advantage of middle grades students' familiarity with, and interest in, M&Ms, concepts of ratio and proportion are developed and applied in this investigation. The candies are used to collect and analyze data, allowing students to generate estimates and predictions. Notions of ratio and proportion are applied by examining the number of colors of M&Ms in one bag and predicting the total number of each color in multiple bags. Equivalent fractions are used to provide a foundation for understanding proportions. Although there are various versions of the M&M experiment (e.g., Corwin and Friel n.d.; Niess 1992), the activities in this investigation are presented to provide some prerequisite experiences with ratio and proportion so that students can apply these ideas in the simulation in Investigation 5. Teachers should decide whether this investigation is necessary to prepare students for Investigation 5.

***With minor modifications, Skittles® may also be used for this investigation.***

After posing and discussing questions about the number of M&Ms in individual bags, the teacher can launch this investigation by posing the question, "What color distribution do you think is intended by the Mars company?"

| | |
|---|---|
| *Prior knowledge:* | Fractions, ratios, equivalent fractions, decimals, percent, mean (arithmetic average) |
| *Concepts:* | Ratio, proportion, equivalent fractions, decimals, percent |
| *Objectives:* | 1. To estimate the number of M&Ms (total number and number of different colors) in a bag (20.9-gram bag or 47.9-gram bag)<br>2. To collect, organize, and analyze data<br>3. To use ratios to compare quantities of different colors of M&Ms<br>4. To use equivalent fractions to develop an understanding of proportions<br>5. To make predictions from data using proportions |

### *Activity 1: M&Ms Melt in Your Mouth—Not in Your Hands!*

*Materials:* 1 bag of M&Ms (20.9 g or 47.9 g) for each student; Activity Sheet 4.1

***This activity uses ratios to compare quantities and make predictions.***

*Procedure:* 1. Hold up one bag of M&Ms and ask students to estimate the number of M&Ms in it. Ask, "Do you think that each bag has the same number of M&Ms? Why or why not? Do you think there is the same number of M&Ms of each color in each bag? Why or why not?" Most students do not expect to find the same number of M&Ms in each bag. Their reasons will vary.

Continue the discussion with these questions: "Do you think the Mars company makes more of one color M&Ms than of any other color? Should we expect to find more M&Ms of one color than of any other color in every bag? Why or why not?"

At this point, teachers may want to introduce or review the term *ratio* by asking students to explain where they have heard it used and what it means to them. Students may use examples related to the colors of M&Ms to explain their thinking.

***A ratio is a relationship between two quantities of the same measure. Examples may include the number of boys as compared to the number of girls in the class; the number of brown M&Ms as compared to the total number of M&Ms.***

2. Distribute bags of M&Ms and Activity Sheet 4.1. Remind students to complete all estimates before counting the M&Ms. Also, remind them not to eat their data until they have completed all their recordings.

***Seventh graders counting and recording M&M data***

3. Review Activity Sheet 4.1 after students complete all the items. Direct students' attention to their answers to items 1, 4, and 5, and compare these estimates to the quantities they wrote in the estimate column at the bottom of the activity sheet. For example, if the estimate in item 1 is 23, is the sum of the estimate column also 23? If brown was the color expected to occur most often in item 4, is the estimate for brown in the table the largest number?

In response to item 9, many students will describe a ratio as "a comparison of two numbers." Although partially correct, this definition is limiting. An interpretation of subtraction involves the comparison of two quantities—for example, if there are twelve brown M&Ms and six red M&Ms, how many more browns are there than reds? Although students do not need to describe a ratio as a multiplicative comparison at this time, they need to recognize the limitations of the definition that states "a ratio is a comparison of two quantities."

Ask, "What kinds of numbers are used to express ratios?" (*Ratios can be expressed as fractions, such as 15/18, which can represent the ratio of boys to girls and is read as 15 to 18. The colon (:) is also used to express a ratio, 15:18.*)

5. Have students describe how they answered item 11, comparing the amount of each color to the total amount. Encourage them to express this relationship as a ratio. These ratios are to be inserted in the last column of the table on Activity Sheet 4.1.

6. Ask students, working in pairs, to compare their data (i.e., their actual counts) and to complete item 12.

*Summary:* After students complete items 9–12 on Activity Sheet 4.1, conduct a class discussion to highlight the "big" ideas of the activity. That is, formulate a working definition of "ratio" (item 9) with several

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

examples (items 10 and 11). Allow time for students to share their comparisons and reactions (item 12).

Ask students to conjecture about which particular colors of M&Ms occur most frequently. They may want to write to Mars, Incorporated, High Street, Hackettstown, NJ 07840, for information about color ratios.

*Assessment:* As students are completing Activity Sheet 4.1, teachers may want to informally assess students' understandings by "spot checking" responses as they walk around the room. This informal assessment may influence instructional decisions. For example, in one grade 7 class, a majority of the students described a ratio (item 9) as "a comparison of two numbers." During the class discussion, the teacher gave examples of other comparisons that were not ratios and asked students to think about how these comparisons differed from ratios. An additive example is, "If I have five marbles and you have two marbles, how many more marbles do I have?" Throughout the entire investigation, the teacher highlighted the nature of ratios in contrast with other comparisons. Another example asked students to find the age of a boy who is twice his brother's age this year but next year won't be.

***Based on the results of consumer preference surveys conducted by the Mars company, the color ratio of M&Ms Plain Chocolate Candies is 30% brown, 20% yellow, 20% red, 10% orange, 10% green, and 10% tan. Since brown is the preferred color because it is indicative of chocolate, brown is the predominant color.***

*Extension 1:* Students may want to analyze the relationship between the number of M&Ms in one bag and the weight of the bag. Why does each bag weigh the same (i.e., 20.9 g), but contain different quantities of M&Ms?

*Extension 2:* Students may want to compare color ratios in small-size (20.9 g) bags with color ratios in larger (47.9 g) bags. After collecting data from several samples, dc they notice any emerging patterns?

***Activity 2: Multiplying M&Ms!***

*Materials:* Data from Activity Sheet 4.1; Activity Sheet 4.2; computers and spreadsheet software (e.g., Microsoft Works) (optional)

***In this activity students use proportions to make predictions.***

*Procedure:* 1. Ask students to use the data they have recorded on Activity Sheet 4.1 to complete the table on the top of Activity Sheet 4.2: Predicting from Data.

2. After students complete the table, provide time to have them explain the strategies they used (item 2).

3. Before assigning item 5, have a discussion about items 3 and 4. The two ratios in each item are equivalent fractions. Set the two ratios in each item equal to each other. For example, Garin, a grade 7 student, found the ratio of brown M&Ms to the total number of M&Ms in one bag to be 8/23. In two bags, the ratio was 16/46. He observed that "the numerator and denominator are multiplied by 2." He found the ratio of red M&Ms to the total number of M&Ms in one bag to be 2/23. In two bags, the ratio was 4/46. He wrote the same observation for this comparison. One of the colors Garin selected for item 5 was yellow. He compared the ratio of yellow M&Ms in one bag to the total number of M&Ms in two bags and three bags. His ratios for single and multiple bags were the following: yellows in one bag, 6/23; yellows in two bags, 12/46; yellows in three bags, 18/69. He noticed that 6/23 = 12/46 = 18/69, recognizing the common factors that yield these equivalent fractions.

4. Many students may not be able to answer item 6, "What is a proportion?" Teachers may want to assign students to research this for home-

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

**Adding these quantities is different from adding fractions. On the one hand, 6/23 (6 out of 23) yellows in one bag plus 6/23 (6 out of 23) yellows in a second bag equals 12/46 (12 out of 46) yellows in two bags. Adding fractions, on the other hand, deals with parts in the same unit. For example, 6/23 of the bag is yellow; 8/23 of the bag is brown; therefore, 14/23 of the bag is yellow or brown.**

work, collecting examples of how the word is used in newspapers, in magazines, or in everyday conversation.

For item 6, Garin wrote, "two ratios that are equal." For item 7 he listed many proportional relationships, one of which was "the number of yellow M&Ms in one bag is proportional to the number of yellow M&Ms in two bags" (6/23 = 12/46).

Students need time to examine the table to look for proportional relationships that exist between the ratios of colors of multiple bags and the total in the same number of bags (e.g., 2/23 = 4/46; the number of red in one bag is to the total in one bag as the number of red in two bags is to the total in two bags) and between the ratios of individual colors in multiple bags (e.g., 2/3 = 6/9; the number of red in one bag is to the number of tan in one bag as the number of red in three bags is to the number of tan in three bags).

*Integrating Technology:* Provide students with the opportunity to set up their data on a spreadsheet. Using such spreadsheet software as Microsoft Works, Visicalc, or Lotus, students can complete the table and analyze the data.

**In this activity, students examine proportional relationships in actual data.**

### Activity 3: Getting a Closer Look in Each Bag

*Materials:* Data from Activity Sheet 4.2; Activity Sheet 4.3; calculators with fraction capability (one for each student); computers and spreadsheet software (e.g., Microsoft Works) (optional)

*Procedure:* 1. Arrange students in groups of four or five to compile their actual data in the table on Activity Sheet 4.3: Group Record Sheet. Review the labels and headings on the table to be sure the students understand the information they will be compiling and analyzing.

2. Depending on the students' ability, teachers may want to review the different ways of expressing rational numbers: fractions, decimals, and percents. Elicit some examples from students (e.g., *1/2 = 0.50 = 50%*). Discuss how these different forms can also be used to express ratios.

3. Depending on their ability, teachers may want students to compute the mean for each color and the total. Allow students to use either the totals or the means to express the ratios in fraction, decimal, and percent forms. Some students may want to compare the data in two ways and then compare the results.

4. Because of the nature of "real" data, students may not find any proportional relationships among the actual data for item 4. However, comparisons between quantities of colors in single bags with quantities of colors in the total may yield some "close" proportional relationships. For example, the ratio of orange M&Ms to the total number of M&Ms in Selana's bag was 2/24 (approximately 0.083), and the ratio of orange M&Ms to the total number of M&Ms in the total of the four students' bags was 8/90 (approximately 0.089). She wrote, "2/24 is almost equal to 8/90."

*Integrating Technology:* Provide students with the opportunity to set up their data on a spreadsheet. Using such spreadsheet software as Microsoft Works, they can find the sums and analyze the data.

*Summary:* Ask, "What color distribution, if any, do you think is intended by the Mars company?"

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

After analyzing the data from the students in his group, Garin found the following color ratios: brown, 35 percent; red, 23 percent; yellow, 18 percent; green, 8 percent; tan, 8 percent; orange, 8 percent. He explained that these are not the ratios listed by the Mars Company but that these results were based on only four bags of M&Ms. He wanted to analyze the data from every student in the class to find out how close the class data compare with the Mars company data. Garin's knowledge of fractions and equivalent fractions, and his facility with factors and multiples, contributed to his success in this investigation. Also, his self-confidence and interest in challenging the Mars data led him to undertake an independent project, analyzing the class's data.

***Students recognize the need to have a large number of data to have "confidence" in results.***

***Closing Comments***

Capitalizing on students' interest and curiosity presents a natural way to introduce and develop rational number ideas. Some educators may not approve of using candy as a means to motivate and generate interest. However, after examining students' level of understanding, the ends may justify the means. If using candy does present a problem, such nonedible objects as marbles, markers, and crayons may be used.

The purpose of this investigation was to give students experience working with ratios and proportions in the context of data collection and data analysis activities. The ideas developed in these activities provide a foundation for building and applying concepts in the simulation for the next investigation.

Name ______________________________ Date ______________

## *M&M EXPERIMENT*

## Individual Record Sheet

1. Estimate the number of M&Ms in one bag. ESTIMATE: __________
   What is the weight of the bag of M&Ms? WEIGHT: __________

2. What colors of M&Ms do you expect to find in one bag? List the colors in the table below.

3. Estimate the number of M&Ms of each color. Write your estimates in the table below.

4. Which color do you expect to find *most* often? ______________________

5. Which color do you expect to find *least* often? ______________________

6. Count the total number of M&Ms in one bag and list the number of each color. Write these quantities in the appropriate spaces in the table below. Do NOT eat your data until you have written all the quantities in the table.

7. Which color appeared *most* often? ________________ How does this compare with your expectation?

8. Which color appeared *least* often? ________________ How does this compare with your expectation?

9. What is a *ratio*?

10. Give three examples of ratios.

11. What is the ratio of the number of each color of M&Ms to the total number of M&Ms? Write each ratio in the table.

12. Using the data you have collected, how many M&Ms would you expect to be in your partner's bag? ____________ Compare your total with your partner's. Are they the same? ____________ Are you surprised? ____________ Why or why not?

| Colors | Estimate of the Number of Each Color | Number of Each Color | Ratio of the Number of Each Color to Total |
|---|---|---|---|
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| Total | | | |

**ACTIVITY SHEET 4.1**

Name ____________________ Date ____________

***M&M EXPERIMENT***

## Predicting from Data

| | Brown | Red | Yellow | Green | Tan | Orange | Total |
|---|---|---|---|---|---|---|---|
| Number in 1 M&M bag (your own) | | | | | | | |
| Number in 2 bags | | | | | | | |
| 3 bags | | | | | | | |
| 4 bags | | | | | | | |
| 5 bags | | | | | | | |
| 6 bags | | | | | | | |
| 7 bags | | | | | | | |
| 8 bags | | | | | | | |
| 10 bags | | | | | | | |
| ⋮ | | | | | | | |
| 100 bags | | | | | | | |

1. Complete the table above, using the data you collected on Activity Sheet 4.1. Using these data, complete the table, identifying how many of each color and the total number of M&Ms you predict for the number of bags indicated.

2. Describe how you completed the table.

3. Make a ratio of the number of brown M&Ms in one bag to the total number of M&Ms in one bag. Make another ratio of the number of brown M&Ms in two bags to the total number of M&Ms in two bags. Compare these ratios. What do you observe?

4. Make a ratio of the number of red M&Ms in one bag to the total number of M&Ms in one bag. Make another ratio of the number of red M&Ms in three bags to the total number of M&Ms in three bags. Compare these ratios. What do you observe?

5. Identify two other colors to compare with the total number of M&Ms in three and four bags. Write the ratios and explain your observations.

6. What is a *proportion*?

7. In the table above, find as many proportional relationships as you can. List them.

***ACTIVITY SHEET 4.2***

Name ______________________________ Date ______________

***M&M EXPERIMENT***

**Group Record Sheet**

1. Working with three or four other students, use the data from your M&M bag and complete the table below. Collect the information from the other students and write it in the spaces provided. Fill in the names of the students in your group.

| Name | Brown | Red | Yellow | Green | Tan | Orange | Total |
|---|---|---|---|---|---|---|---|
| 1. | | | | | | | |
| 2. | | | | | | | |
| 3. | | | | | | | |
| 4. | | | | | | | |
| 5. | | | | | | | |
| Total | | | | | | | |
| Mean | | | | | | | |

Ratio of color total to total number in

Fraction Form

Decimal Form

Percent Form

2. Find the ratio of the total number of each color to the total number of M&Ms in all the bags. How many bags of M&Ms were used to complete the information in the table above? __________ Compare the totals and ratios with the prediction for this number of bags in the table on Activity Sheet 4.2. Were the actual data the same as your predictions? __________ Why or why not?

3. What is a *proportion?*

4. Identify any proportional relationships in the data listed in the table above.

***ACTIVITY SHEET 4.3***

## INVESTIGATION 5: SHARKS IN A BOX: A TAGGING-AND-RECAPTURING SIMULATION

*Frances R. Curcio and J. Lewis McNeece*

Many situations exist in which scientists employ rational number concepts for estimations and predictions based on data collected from a representation of a population. For example, in the videotape, *Estimation: Am I Close?* from the *Challenge of the Unknown* (Crimmins 1986), marine biologist Samuel Gruber from the University of Miami is studying the growth, movement, maturation, and reproduction of lemon sharks. He uses ratios and proportions to estimate the number of lemon sharks in Bimini Lagoon. One statistical technique that he uses involves tagging sharks and then recapturing them. Using this technique, he can get an estimate of the number of sharks in the lagoon by comparing the number of tagged sharks he recaptures to the total number of sharks he captures.

Showing the video clip before the simulation builds the foundation for posing such questions as, How many sharks are in the lagoon? How can we estimate the number of sharks using ratios and proportions?

*Prior knowledge:* Fractions, ratios, equivalent fractions, decimals, percent, mean (arithmetic average)

*Concepts:* Ratio, equivalent fractions, proportion

*Objectives:*

1. To estimate the number of cubes in a shoebox
2. To collect, organize, and analyze data
3. To use equivalent fractions to develop an understanding of proportions
4. To estimate the total number in a population by using proportions and extrapolating from a sample

### *Activity 1: These Sharks Don't Bite!*

*Materials:* "Sharks" video clip from *Estimation: Am I Close?* (*The Challenge of the Unknown* 1986); video playback equipment; Activity Sheet 5.1; a shoebox for each group of four or five students filled with different quantities (150–250) of centimeter cubes and labeled on the cover with a different letter to facilitate recording and comparing the number of cubes in each box; one package of 450 self-adhesive circular color-coding labels (6 mm in diameter)

***In this activity, students appreciate the "power" of a simulation in which sharks are represented by cubes and the lagoon is represented by a shoebox.***

*Procedure:* 1. Give the following background information for viewing the four-minute video clip on estimating the number of sharks in Bimini Lagoon: Samuel Gruber is a marine biologist at the University of Miami who studies lemon sharks. In the film, Gruber struggles with the problem of determining how to estimate the number of sharks in Bimini Lagoon (Maddux 1986). Pose this question, "What are some ways that you think he can estimate the number of sharks in the lagoon?"

***"The Challenge of the Unknown" videotape series is available free from Karol Media, P.O. Box 7600, Wilkes-Barre, PA 18773.***

2. Show the video clip.

3. Discuss the methods employed by Gruber—tagging and recapturing and the aerial survey.

4. Introduce the idea of a simulation. When a task is too expensive, too dangerous, or too time-consuming, scientists design simulations to model the real situation. Describe how students will simulate tagging

**Seventh graders tagging and recapturing "sharks"**

and recapturing sharks by using a shoebox to represent the lagoon, the cubes to represent the sharks, and the circular coding labels to represent the tags.

5. Distribute shoeboxes containing the cubes and Activity Sheet 5.1: Tagging and Recapturing "Sharks." Students should be expected to complete items 1–4.

*Summary:* 1. Review items 1–4. Ask students to describe the strategies they used to estimate the number of cubes in the box.

2. If time permits, challenge students to think of ways to find *N* (see item 5).

**Activity 2: Haven't I Seen This Shark Before?**

*Materials:* Activity Sheet 5.1; data from the tagging and recapturing experiment in Activity 1

*Procedure:* 1. In preparation for setting up and solving proportional expressions, ask students to find *N* to make these equations true statements:

a. $\frac{3}{N} = \frac{1}{2}$ $(N = 6)$ b. $\frac{7}{N} = \frac{2}{8}$ $(N = 28)$ c. $\frac{12}{N} = \frac{25}{100}$ $(N = 48)$ d. $\frac{3}{N} = \frac{17}{38}$ $(N = 6)$

Are some of these equations easier to solve than others? (*The equations* a, b, *and* c *are easier than* d.) Why? (*Fractions in* a, b, *and* c *can be simplified to unit fractions, and the equivalent fraction can then be determined by multiplying the numerator and the denominator by the same form of the identity element, i.e.,* a. *3/3*, b. *7/7*, c. *12/12*.) To solve for *N* in (d), ask students to estimate a unit fraction that is "close" to 17/38 (*1/2*). Then ask them to estimate the value of *N* (*approximately 6*).

**In this activity, students use proportions to predict from data.**

2. Discuss ideas for using information about *T* (the number of sharks tagged), *t* (the number of tagged sharks recaptured), and *n* (the total number of sharks captured) to find *N* (item 5, the total number of sharks in the lagoon). Ask students to describe the relationship of *t/n* (the ratio of the number of tagged sharks recaptured to the total number of sharks captured). Ask students to identify another relationship that may have a similar ratio (*T/N*, the number of sharks tagged at the beginning of the experiment to the total number of sharks in the lagoon). We assume that these two ratios, *t/n* and *T/N*, are equal. Students should record the proportional relationship:

$$\frac{t}{n} = \frac{T}{N}$$

Ask, "Does this relationship make sense? Why or why not?"

3. Select data from students in the class to illustrate a proportional relationship. For example, in one grade 7 class, $T = 25$ for group A. Alina's handful yielded $t = 3$, and $n = 18$. Then, $25/N = 3/18$. Students were

asked to identify a unit fraction to use to generate an equivalent fraction with 25 in the numerator. That is, $3/18 = 1/6$; so $1/6 = 25/N$. $25(1) = 25$; $25(6) = N$; $N = 150$. Ask, "What does 150 mean?"

Leslie was also in group A, where $T = 25$. Her handful yielded $t = 2$ and $n = 30$. Then, $2/30 = 25/N$. $2/30 = 1/15$; so, $1/15 = 25/N$. $N = 15(25) = 375$.

Ask the students what happens when $t/n$ cannot be simplified to a unit fraction. For example, in group A, where $T = 25$, John's handful yielded $t = 2$ and $n = 27$. $2/27$ cannot be simplified, but it is close to $1/13$. $N$ is approximately $25(13) = 325$.

4. Ask students to find the mean for $N$ (item 5 on Activity Sheet 5.1). For example, with the data from Alina, Leslie, John, and Kerri, the mean for $N$ in group A is 272, which is a "good" estimate for the number of cubes in the box—255.

*Summary:* 1. Have students complete items 6, 7, and 8. For example, in group A, the number of cubes in the box was 255. The greater the number of recaptures, the closer the estimate of $N$ is to the actual count.

2. What is a proportion? How are proportions related to equivalent fractions? How can proportions be used to estimate the size of a population?

*Extension:* Students may want to estimate the number of dogs or cats in their neighborhood using a modification of this tagging-and-recapturing technique. They should consider safe ways to tag and recapture the animals so that they do not harm the animals or put themselves in any danger. For example, students may want to survey a small number (perhaps five) of blocks, counting the number of dogs or cats seen in a given period. On the basis of the number counted, students can make an estimate of the total number of dogs and cats (Maddux 1986, p. 203).

### Closing Comments

This investigation supplies a context in which a real-world problem is analyzed using a simulation that employs rational number and proportion concepts. Students develop and apply these concepts in a meaningful way by collecting and analyzing data. Teachers should furnish other opportunities to reinforce these concepts and embed proportions in a variety of situations (Lesh and Lamon 1992). By varying the situations, concrete examples of these ideas can be generalized and abstracted. In these cases, the teacher acts as an orchestrator, asking students to describe similarities across situations and contexts. A similar tag-and-recapture activity can be found in *Mathematical Modeling in the Secondary School Curriculum* (Swetz and Hartzler 1991).

Name ____________________________ Date ______________

**EXPERIMENT: TAGGING AND RECAPTURING "SHARKS"**

1. Remove the cover of the shoebox, look in, and estimate the number of cubes in the box. Record the letter that appears on the cover of the box.

ESTIMATE: ________________ BOX LETTER: ___________

2. Write the name of each student in your group in the table below.

3. Have one person in your group take out a handful of cubes. Tag each cube with a circular label. Count the number of cubes tagged and record this amount under "*T*" in the table below. This amount will be the same for each member in your group. Tagging is to be done **only once.**

4. Let the sharks take a "swim" (shake up the cubes in the box!). One at a time, each person in your group should remove a handful of cubes. Keep a record of the total of each person's "catch" by writing the number of cubes in that catch in column "*n*." Count the number of tagged cubes and write this number in column "*t*." Replace all cubes in the box. Continue this process until everyone has had a chance to capture sharks and record the information in the table.

| Name | *T* | *t* | *n* | *N* |
|---|---|---|---|---|
| 1. ___________ | ______ | ______ | ______ | ______ |
| 2. ___________ | ______ | ______ | ______ | ______ |
| 3. ___________ | ______ | ______ | ______ | ______ |
| 4. ___________ | ______ | ______ | ______ | ______ |
| 5. ___________ | ______ | ______ | ______ | ______ |
| MEAN | ______ | ______ | ______ | ______ |

$T$ = number of sharks (cubes) tagged
$t$ = number of tagged sharks (cubes) recaptured in a second and subsequent samples
$n$ = total number of sharks captured (total number of cubes in the second and subsequent samples)
$N$ = total number of sharks in the lagoon (cubes in the box)

5. How can you use the information collected by everyone in your group to predict the number of sharks in the lagoon? If you know *T*, *t*, and *n*, how can you find *N*? Each student should compute *N* and record its value in the table. Then, find the mean for *N*.

6. How does your estimate compare with the result determined by finding the mean for *N*?

7. Count and record the number of cubes in your box.

ACTUAL COUNT: ____

8. How does your estimate compare with the actual count? How does the mean for *N* compare with the actual count?

**ACTIVITY SHEET 5.1**

# CLUSTER C: MAKING RATIONAL NUMBER CONNECTIONS WITH SIMILARITY

This cluster contains two investigations: Trips to Worlds of Shadow Creatures and What in the World Are Fractals? Connections among geometry, similarity, ratio, and proportion illustrate the interrelatedness of mathematics.

In the first investigation, students use rational number concepts to create similar shapes and compare the perimeter, the area, and the volume of similar shapes. Extensions that build on the concepts developed and the interest generated integrate modern art, children's literature, movies, and amusement parks. These suggested activities enhance students' appreciation of the presence and utility of rational numbers.

Fractal geometry is a relatively new development in modern mathematics. Rational numbers serve a new role in defining fractional dimension. In the second investigation, as students determine the fractional dimension of shapes and create self-similar shapes, they use notions of rational numbers and ratio. As a culminating activity, students are challenged to create their own fractals.

## INVESTIGATION 6: TRIPS TO WORLDS OF SHADOW CREATURES

*Alice F. Artzt and Claire M. Newman*

An important component of formal thought acquired during adolescence is proportional reasoning (Hoffer and Hoffer 1992, p. 303). Designing activities that promote the recognition of, and discussion about, proportional situations will provide students with the opportunity to build a foundation on which more formal structures can develop. The complexities of understanding proportional relationships must be considered when developing activities for middle school students.

The four activities in this investigation allow students to create, examine, and analyze similar shapes as they explore ways to solve problems that require proportional reasoning. Suggestions for extensions that integrate modern art (the work of René Magritte), children's literature (*Gulliver's Travels*), movies (*Honey, I Shrunk the Kids* and *Honey, I Blew Up the Kid*), and amusement parks (Fievel's Playland), are included.

*Prior knowledge:* Perimeter, area, and volume; equivalent fractions; ratio and proportion; rectangular coordinates

*Concepts:* Ratio, proportion, similarity

*Objectives:*

1. To extend notions of equivalent fractions to proportional relationships
2. To create shapes that are similar
3. To compare the perimeter, area, and volume of similar shapes
4. To analyze similar shapes depicted on a rectangular grid

*Problem:* How can we build a world for a shadow that comes to life?

### Formulating the Problem

Carlos and his friends have been asked to make a mural for a large wall in their school. They decide to use an old movie projector to project

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

Carlos's shadow onto the wall. The students notice that the shadow looks just like Carlos, but it is absolutely tremendous. They trace the outline of the shadow onto the wall. Carlos is only 120 cm tall and the shadow is 240 cm tall. Carlos and his friends decide to decorate the wall by painting the shadow and putting ribbon around the outline.

A week before, the students in Carlos's class traced outlines of their bodies on large pieces of paper, putting ribbon around the outline and painting the interior. Carlos used six meters of ribbon to outline his body and five small jars of paint to fill it in. How much ribbon will be needed to outline the shadow? How many jars of paint will be needed to color in the shadow? Give students the opportunity to discuss their conjectures. Record some of these on the board or on a transparency.

**Activity 1: Getting to Know the Shadow**

*Materials:* Pattern blocks, one ruler, and a copy of Activity Sheet 6.1 for each student

*Procedure:* 1. To help students get a better understanding of the problem setting, have them model it using pattern blocks. A pattern block drawing of Carlos is on Activity Sheet 6.1. Students are to use pattern blocks to create a figure that is an enlargement of the given one. It must have the same shape as the original and be twice as tall and twice as wide. Students should work on this in pairs. The figures should look like those in figure 6.1. Point out that since the two figures have the same shape, they are said to be *similar*.

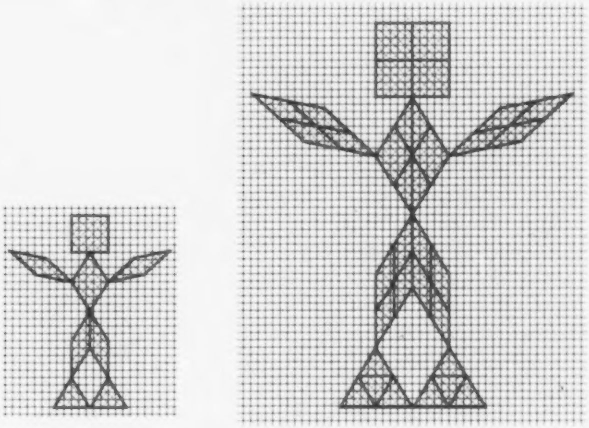


**Fig. 6.1**

2. Ask students to measure the width of the head of the smaller figure and that of the head of the larger one, the length of the smaller arm and that of the larger arm, as well as other corresponding parts to show that the ratio of the longer length to the corresponding shorter length in each case is 2:1, or 2/1. Introduce the meaning of *scale factor*. Since each linear measurement of the larger figure is twice that of the smaller one, we say that the smaller figure has changed by a *scale factor* of 2.

3. Using the edge of a pattern block as one unit, have students compare the number of units around the outside of the smaller figure with the number of units around the outside of the larger figure. Ask students to compare the number of shapes used to construct the smaller figure with the number used to construct the larger one. How many squares for each? How many triangles? How many large rhombi? How many small rhombi?

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

4. Ask students to reconsider the original problem setting and make conjectures regarding the length of ribbon and the number of square centimeters (or the number of jars of paint) needed.

5. Help students realize that what makes the problem difficult is the complex shape of the human body. Investigations with a simpler shape will suggest relationships between Carlos's shape and his shadow. Ask students to approximate the shape of the body with a rectangle. By examining pairs of rectangles (with each rectangle in the pair a different size), students can notice patterns that will help them support or refute their original conjectures.

### *Activity 2: Getting to Know the Shadow—Inside and Around*

*Materials:* Activity Sheet 6.2 (one for each student)

*Procedure:* 1. Help students recognize that the number of meters of ribbon needed is determined by the perimeter of Carlos's shadow and the number of jars of paint needed is determined by the area of the shadow.

2. In order to reinforce the ideas of similarity and ratios of corresponding linear measurements, have students work in small groups to explore pairs of rectangles using Activity Sheet 6.2. After studying various pairs of rectangles, students should conclude that when corresponding measurements have the same ratio, the rectangles are similar. That is, when length of $A$/length of $B$ = width of $A$/width of $B$ or length of $A$/width of $A$ = length of $B$/width of $B$, rectangle $A$ is similar to rectangle $B$. Also, the ratio of perimeters of similar rectangles is equal to the ratio of their lengths or widths; in this example, perimeter of $A$/perimeter of $B$ = length of $A$/length of $B$ = width of $A$/width of $B$. Note that length refers to the longer side. Notice also that the rectangles in question 4 are similar and the ratio of corresponding linear measurements is 1:1. Point out that when the corresponding parts of similar figures have the ratio 1:1, the figures are said to be *congruent*.

3. In Activity Sheet 6.2, students are given pairs of similar rectangles, and they compare the ratio of the areas with the ratio of the meaurements of corresponding sides. Students should conclude that when the ratio of the corresponding sides is $A/B$, the ratio of the areas is $A^2/B^2$. Students can now begin to speculate about the area of the shadow in relation to Carlos's body shape. They are now ready to answer this question: How many square centimeters (or jars of paint) will be needed to cover the giant shadow if the shadow is twice as tall and twice as wide as the original body shape? Will twice as much paint be enough?

*Summary:* Since the shadow figure is similar to Carlos's body outline and the ratio of the linear measurements is 2:1, we may agree that the ratio of the perimeter of the shadow to the perimeter of Carlos's body shape is 2:1. Now students can find the length of ribbon needed to outline the shadow by multiplying the length of ribbon needed for Carlos's body outline by 2. Since Carlos needs six meters of ribbon for his original outline, the outline of his shadow will require twelve meters of ribbon ($2 \times 6 = 12$).

It is now also possible to conclude that the ratio of the area of the shadow to the area of Carlos's body shape is $2^2:1$, or 4:1, and twenty jars of paint will be needed to paint Carlos's shadow ($4 \times 5 = 20$).

### *Extension 2.1: Similar Figures*

Have students enlarge and reduce photographs or drawings on a photocopy machine that has a variable enlargement and reduction feature.

They can then compare lengths, widths, perimeters, and areas of similar figures and draw conjectures about the relationships. For related ideas see Artzt (1991) and Reys (1991, p. 7).

***Extension 2.2: Proportions***

Have students experiment with creating similar rectangles. They can be guided in this exploration by furnishing them with a given rectangle and an incomplete rectangle that must be drawn similar to the given one. Challenge them to find more than one solution. In some cases it may be advisable to use proportions. Rectangles such as those drawn in figure 6.2 can be used.

***Identifying patterns is the essence of inductive reasoning. Students can often consider or generate a set of specific instances, organize them, and look for a pattern. (NCTM 1989, p. 82)***

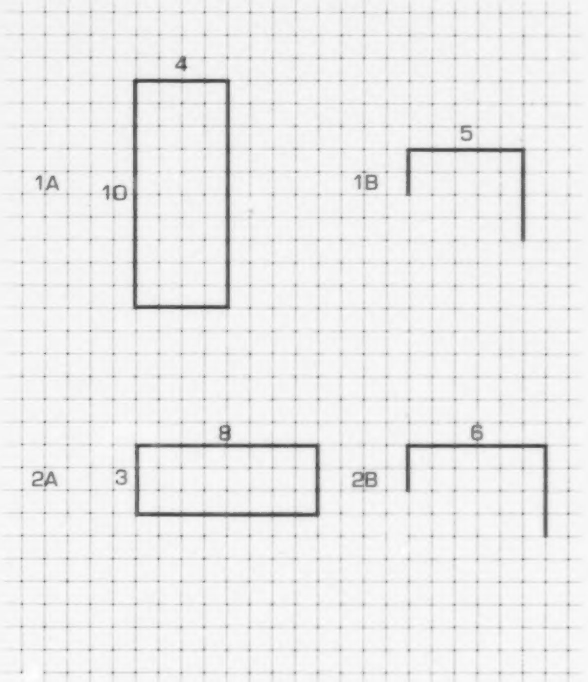
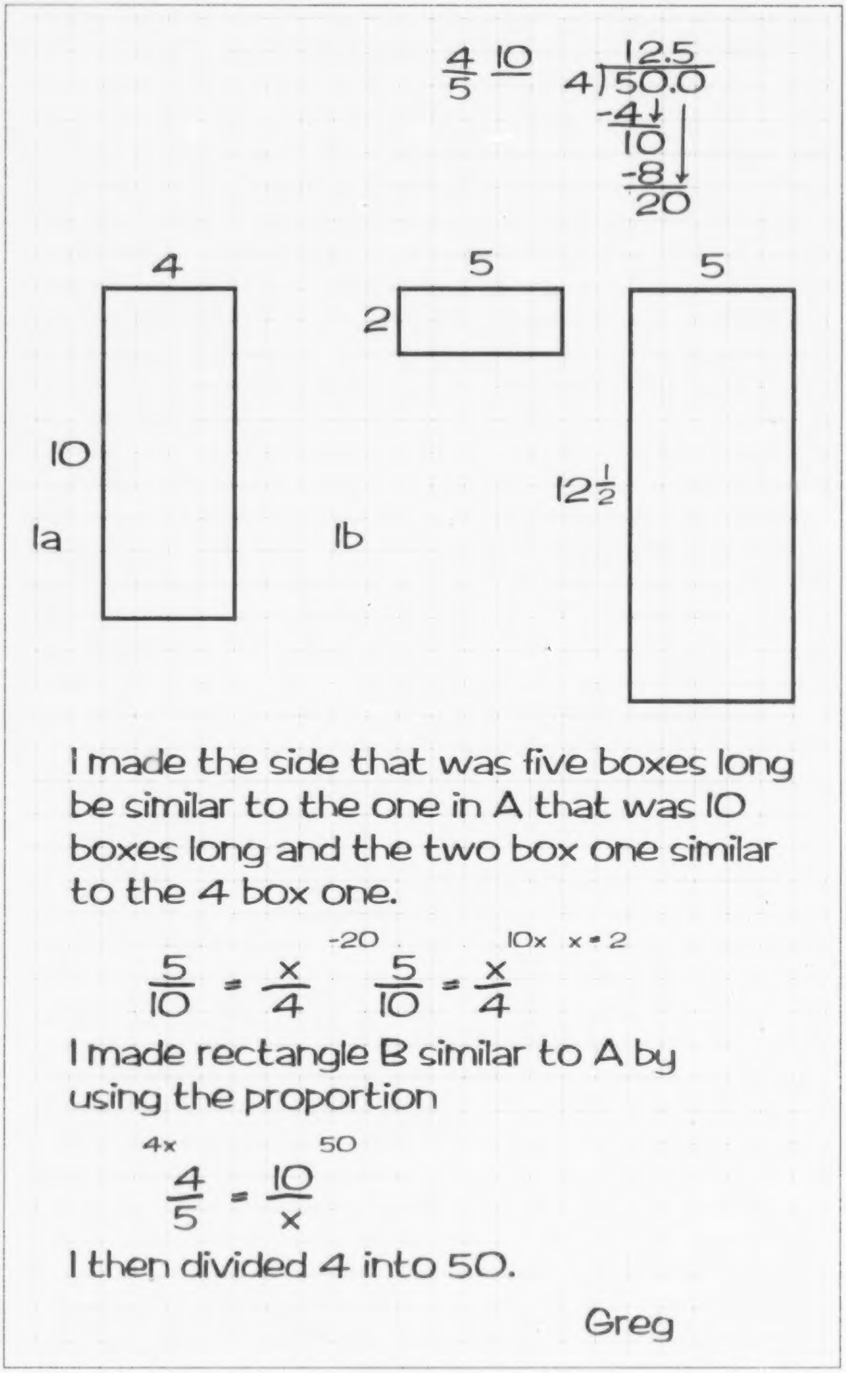


***Fig. 6.2***

Notice in these examples that there is no indication of whether the given side of rectangle B is its length or its width. In example 1, if the length ($L$) of rectangle B is 5 units, then for the fractions 4/10 and $W/5$ to be equivalent, the width ($W$) of rectangle B would have to be 2 units. In other words, 4/10 = 2/5. However, if the *width* of rectangle B is 5 units, then we must find the length, such that $4/10 = 5/L$. Students who know how to solve proportions will employ a variety of methods (such as equivalent fractions or unit rate; see Introduction, p. 4) to conclude the length equals 12 1/2 units. Similarly in example 2, either $3/8 = 6/L$ and $L = 16$, or $3/8 = W/6$ and the width equals 2 1/4 units. In each example, students should be asked to find the perimeters and verify that the ratio of the perimeters is the same as the ratio of any pair of corresponding sides.

*Assessment:* By having students give written descriptions of their work, teachers become aware of specific skills that need reinforcing or possible misconceptions that need correcting. For example, in figure 6.3, despite the fact that Greg solved the first problem in figure 6.2 correctly, his written explanation reveals several areas of misunderstanding. He is confused about the mathematical meaning of *similar*. He needs to recognize that rectangles are similar, but their sides *correspond*.


4
10
1a
5
2
1b
5
$12\frac{1}{2}$

$$\frac{4}{5} \quad \frac{10}{} \qquad 4\overline{)50.0}^{\;12.5} \quad \begin{array}{r} -4\downarrow \\ \hline 10 \\ -8\downarrow \\ \hline 20 \end{array}$$

I made the side that was five boxes long be similar to the one in A that was 10 boxes long and the two box one similar to the 4 box one.

$$\frac{5}{10} = \frac{x}{4} \qquad \overset{-20}{} \quad \frac{5}{10} = \overset{10x \quad x = 2}{\frac{x}{4}}$$

I made rectangle B similar to A by using the proportion

$$\overset{4x}{\frac{4}{5}} = \overset{50}{\frac{10}{x}}$$

I then divided 4 into 50.

Greg

**Fig. 6.3**

Furthermore, it is important for the teacher to reinforce the idea that "boxes" can be used as measures of area but *not* as units of length.

**Extension 2.3: Similarity on a Coordinate System**

*Materials:* Rectangular grid paper; Activity Sheet 6.3

In order to develop a better understanding of the concepts of similarity and proportion, students should conduct investigations in different settings. The following activity is an example of how a rectangular coordinate system can be used to make connections with similarity and ratio.

Let us return to Carlos's shadow. Imagine that the pattern for a shirt Carlos wears is drawn on a coordinate system as in Activity Sheet 6.3. After the students complete each part of the activity, ask them to determine whether they have produced a pattern suitable for the shadow. Discuss which of the shirts is similar to Carlos's. For the similar shirts, ask students to verify that the corresponding sides have the same ratio. The students should compare the perimeter of the shadow's shirt with that of Carlos's shirt. Students can also count the number of squares in the interior of the figures and then compare the ratio of the areas. Related activities can be found in *Similarity and Equivalent Fractions* (Lappan et al. 1986).

*Cooperative Learning.* The task of Activity Sheet 6.3 lends itself to problem solving in small groups. To reduce the time spent on computation and to obtain each graph as rapidly as possible, the work can be divided among three pairs of students. Each pair should be responsible for obtaining the coordinates of the points and the graph of one shirt pattern. When this task has been completed, the group of six can meet to discuss their results and decide which pattern would be suitable for the shadow creature.

### *Extension 2.4: Reptiles and Areas*

*Materials:* Pattern blocks; rectangular and triangular grid paper

In order to develop a more thorough understanding of the growth of area in similar figures, students can study the sequence of square numbers from a geometric perspective. By returning to the students' pattern block representations and their shadows (or Carlos and his shadow), students can see how shapes (the blocks) are repeated to create a larger, similar body part. Whenever it is possible to create a figure similar to a given block (or tile) by repeating that block (or tile), the figure is called a *reptile*. For further information about reptiles see *Similarity and Equivalent Fractions* (Lappan et al. 1986).

Guide students in using their pattern block shapes to create the reptiles shown in figure 6.4. To simplify drawing the figures, they can use rectangular and triangular grid paper. Students should organize their results in a table similar to the one in the figure so that they can observe various relationships. They should see that the number of blocks (or tiles) needed to form each figure is a square number. When these similar figures are arranged in order of size, their areas form the sequence of square numbers 1, 4, 9, ....

### *Activity 3: Stepping Out with the Shadow*

*Problem:* Imagine that the shadow comes to life! It steps out of the wall and becomes a giant who looks just like Carlos. The shadow creature is not only twice as tall as Carlos and twice as wide, it is also twice as "thick." If Carlos weighs thirty kilograms, about how much does the shadow creature weigh?

*Materials:* Unit cubes; one calculator with fraction capability and a copy of Activity Sheet 6.4 for each student

*Procedure:* 1. Ask students to decide whether the shadow creature weighs twice as much as Carlos weighs, four times as much, or more. Students should recognize that its thickness as well as its height and

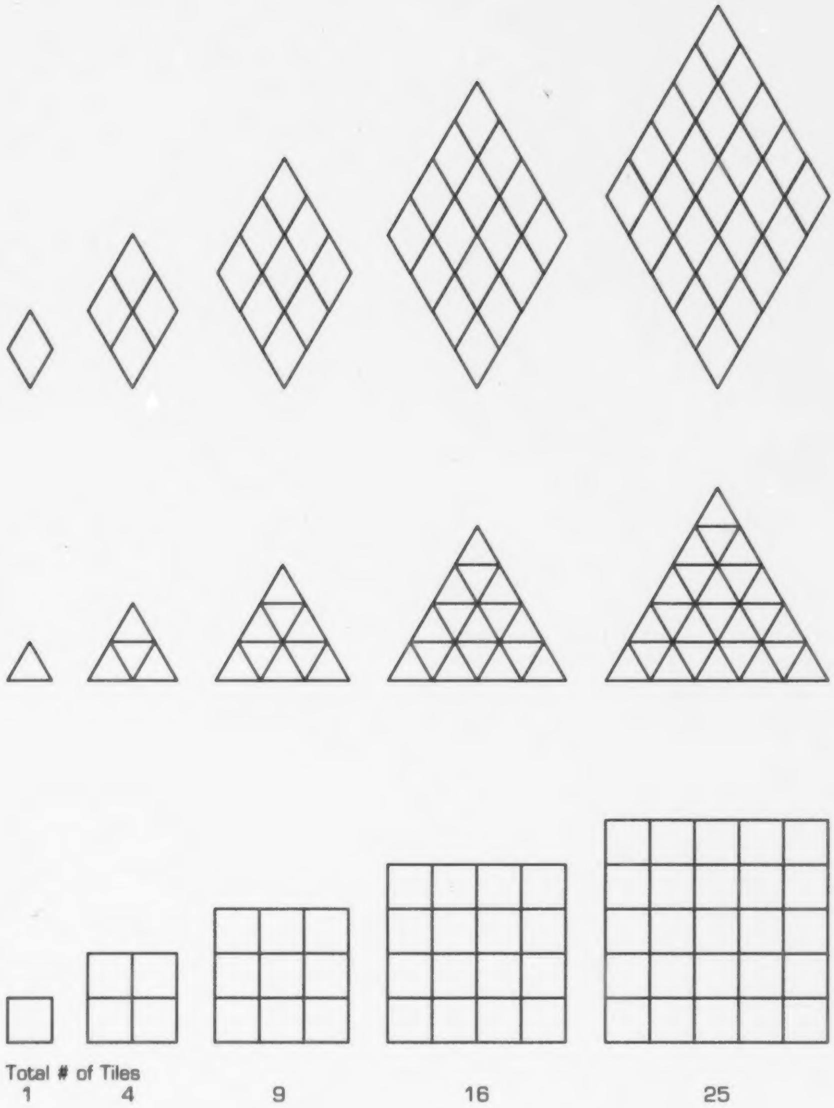


Fig. 6.4

width affect what the shadow creature weighs. This fact suggests that an investigation of volume will supply clues to the solution of the problem. Once again, it will be helpful to use simpler shapes to investigate volume. Ask students to use a rectangular solid to represent a body, beginning with the simplest case—a cube. They should use cubes to build a rectangular solid with all three dimensions twice that of the given solid (see fig. 6.5). Have them find the volume of the larger solid and speculate further on how much the shadow creature weighs.

Activity Sheet 6.4 allows students to explore further the volumes of similar rectangular solids. A pair of rectangular solids are said to be similar if their corresponding lengths, widths, and heights have the same ratio. Students may wish to build some of the rectangular solids with cubes to confirm their calculations of volume. It may be helpful to have students work in small groups so that they can divide up the work required in

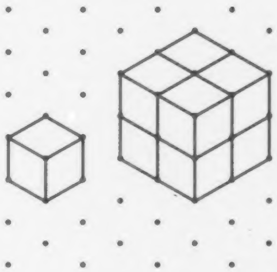


Fig. 6.5

Activity Sheet 6.4. A calculator with fraction capability can facilitate the required simplification of fractions.

*Summary:* After studying pairs of similar rectangular solids, students should conclude that if the ratio of corresponding linear measurements is $a/b$, the ratio of their volumes is $a^3/b^3$.

By the same reasoning, if the ratio of the shadow creature's height to Carlos's height is 2/1, the ratio of their volumes is $2^3/1^3$, or 8/1. Therefore, the shadow creature weighs eight times what Carlos weighs—240 kg (3 × 30 = 240).

**Extension 3.1: Surface Area**

*Problem:* Carlos's total skin area measures 15 000 square centimeters. How many square centimeters of total skin area will the shadow creature have?

*Procedure:* 1. Help students recognize that to solve this problem they must consider Carlos's surface area and that of the shadow creature. Suggest that students use the models constructed earlier to determine that the ratio of the surface areas is $2^2:1^2$, or 4:1. Further study of the surface areas of similar rectangular solids should reveal that if the ratio of the corresponding linear measurements is $a/b$, the ratio of the surface areas will be $a^2/b^2$.

By the same reasoning, if the ratio of the shadow creature's height to Carlos's height is 2/1, the ratio of their surface areas is $2^2/1^2$, or 4/1. Therefore, it would take 60 000 square centimeters of skin to cover the shadow creature's body (4 × 15 000 = 60 000).

**Extension 3.2: Patterns of Growth: Area and Volume**

*Problem:* Carlos can lift a friend his own size. Can the shadow creature lift another shadow creature that weighs as much as it does?

*Materials:* Rectangular grid paper; a transparency of figure 6.3 (optional)

Students have discovered, through previous activities with similar figures, that linear measurements grow by constant factors, area grows by square numbers, and volume grows by cube numbers. According to Haldane (1956), a person's strength depends on the cross section of his or her muscle. Therefore, just as a student's ability to lift his or her friend depends on the area of a cross section of his or her muscle, the shadow creature's ability to lift another shadow creature depends on the area of the cross section of *its* muscle. The area of the shadow creature's cross section would be $2^2$, or 4, times the area of the student's cross section. This means that the shadow creature could lift something only four times as heavy as Carlos. But the shadow creature would be $2^3$, or 8, times as heavy as the student. Thus, it would not be able to lift another shadow creature its own size.

> ***"Identifying patterns is a powerful problem-solving strategy. It is also the essence of inductive reasoning. As students explore problem situations appropriate to their grade level, they can often consider or generate a set of specific instances, organize them, and look for a pattern" (NCTM 1989, p. 82).***

Students might enjoy viewing the videotape, *Similarity* (Apostol 1990), to learn about the limitations of a giant grasshopper. Other ideas related to this can be found in *Measurement in the Middle Grades* (Geddes 1994).

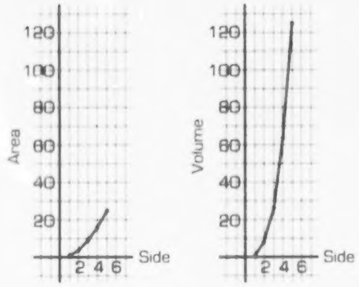
To help students appreciate the growth patterns of area and volume, have them construct graphs. Students should calculate the length of a side and the area of squares and the length of a side and the volume of cubes. By constructing the two corresponding graphs, the students will observe that area grows by square numbers, whereas volume grows by cube numbers (see fig. 6.6). You may wish to make transparencies of the graphs and superimpose one on the other.

**Concluding Assessment Activities**

1. Ask students to list items in their (or Carlos's) world and decide how they would compute the items' respective sizes in the shadow creature's world. Examples of items they may wish to explore are size of an eyeball, waist size, diameter of a circular cookie, size of a footprint, size of a bottle of soda in liters, room dimensions, and so on. Students should be asked to justify multiplication by 2 (linear dimensions), by 4 (area), or by 8 (volume). You can use this activity to assess students' understanding of the principal ideas of this investigation.

2. To assess further students' understanding of similarity and ratio, you may wish to have students view the films *Honey, I Shrunk the Kids* and *Honey, I Blew Up the Kid* and address such questions as these: How were the children made to look so small or so large? How were ratio and proportion involved in making the props used in each film? For further ideas on how these films can be used in other related activities, see *Measurement in the Middle Grades* (Geddes 1994).

3. The painting in figure 6.7, titled *Personal Values,* by modern artist René Magritte (1898-1967), offers the opportunity for students to do a project on size and ratio. They should consider such questions as these: If the comb were life-sized, how large would the bed be? If the bed were life-sized, how large would the comb be? As an added challenge, they could create their own pictures with puzzling ratios.



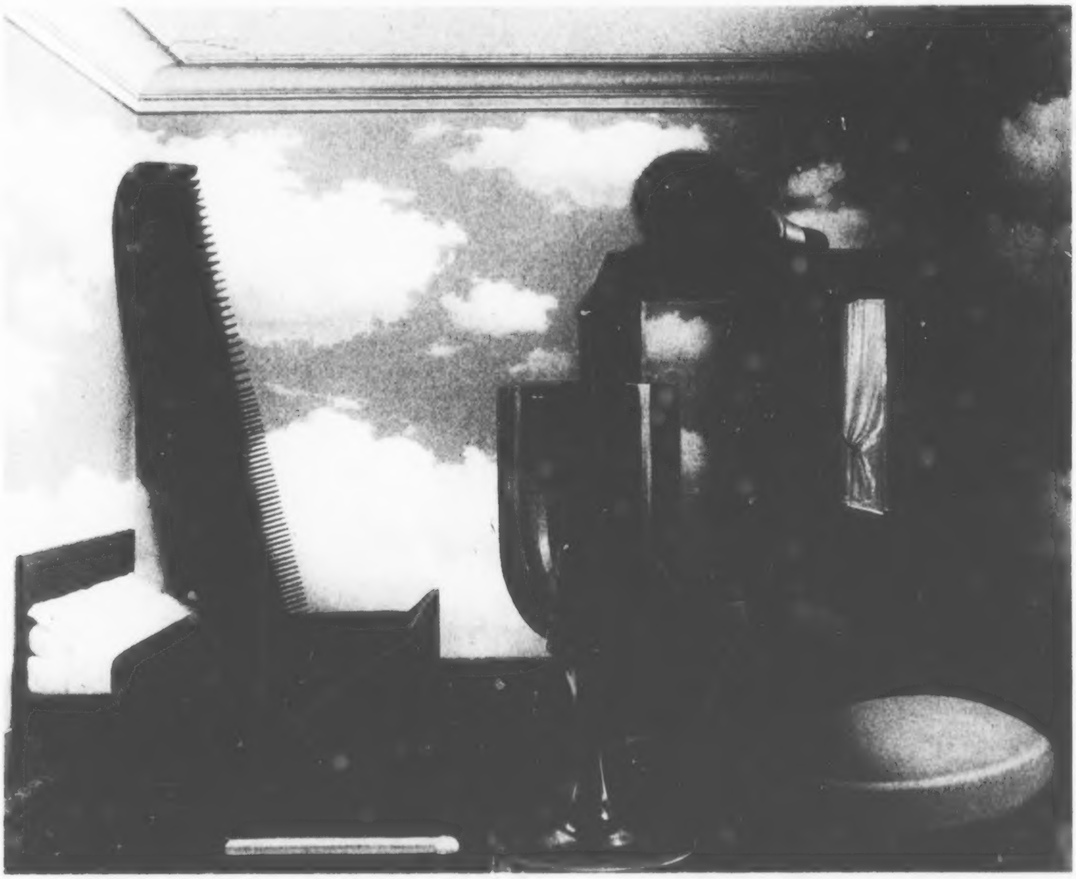
Square

| Side | Area |
|---|---|
| 1 | 1 |
| 2 | 4 |
| 3 | 9 |
| 4 | 16 |
| 5 | 25 |

Cube

| Side | Volume |
|---|---|
| 1 | 1 |
| 2 | 8 |
| 3 | 27 |
| 4 | 64 |
| 5 | 125 |

**Fig. 6.6**



© 1992 C. HERSCOVICI/ARS, New York

**Fig. 6.7**

BEST COPY AVAILABLE

4. For additional experiences related to similarity, students can read (or review) *Gulliver's Travels* (Swift 1992). Have them analyze the size of the Lilliputians in comparison to Gulliver. The "Rule of Thumb" used by the Lilliputians to make new clothes for Gulliver could also be examined. A similar activity, *My Travels with Gulliver* (Educational Development Center 1991), takes students through the "worlds" Gulliver encountered.

5. On 2 July 1992, a new attraction, Fievel's Playland, opened at Universal Studios Florida in Orlando (see fig. 6.8). Find the height of the cowboy who can wear the 1 000-gallon cowboy hat.

© Universal Studios Florida (1992). All rights reserved. Fievel and Fievel's Playland are registered trademarks of Universal City Studios, Inc., and Amblin Productions.

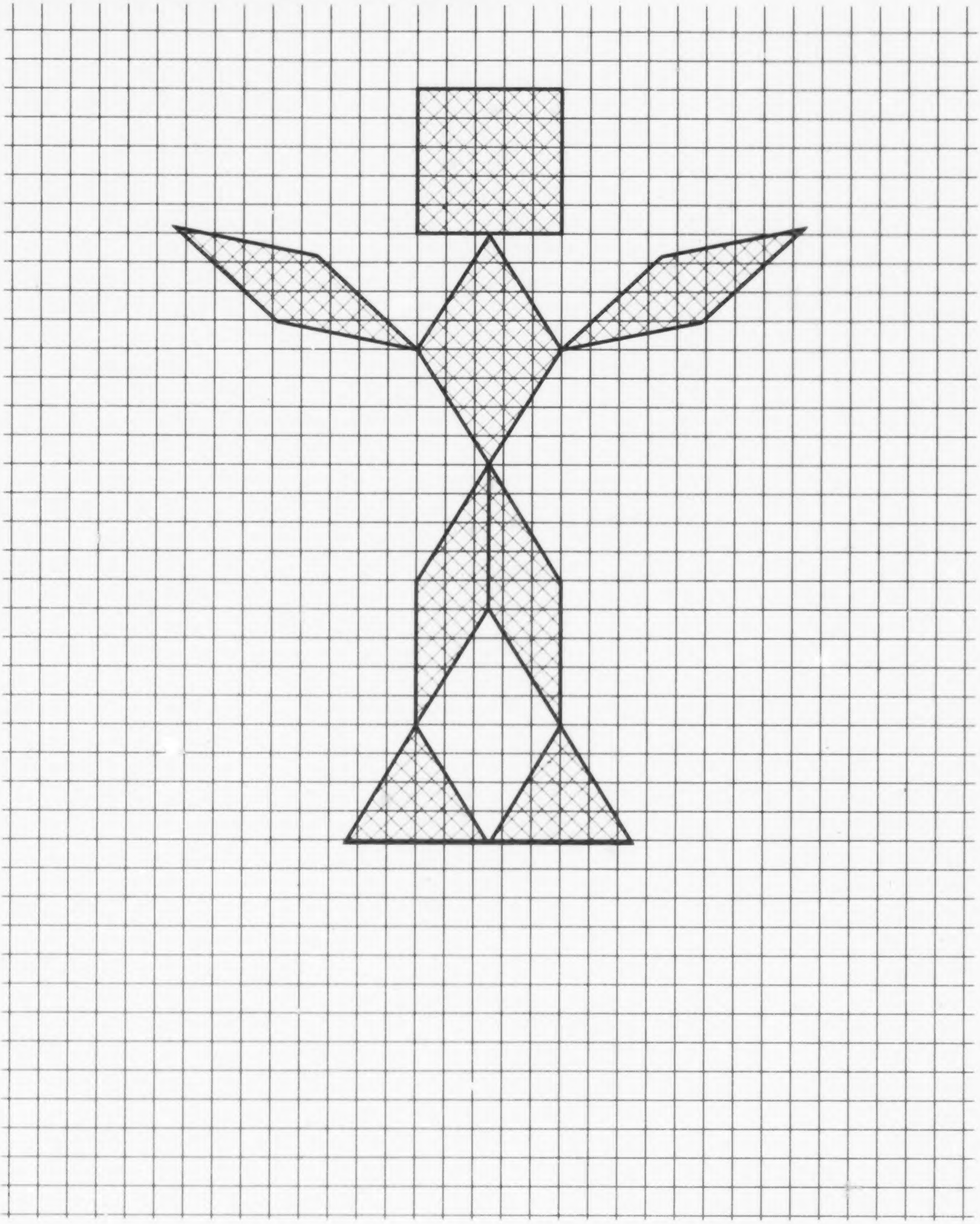
***Fig. 6.8***

***Closing Comments***

The primary purpose of these activities is to show the connections between similarity, ratio, proportion, and equivalent fractions. By creating a giant shadow creature and then modeling it with rectangles and rectangular solids, the activities encourage exploration of perimeter and area of similar rectangles and the volume of similar rectangular solids. It is through these activities that students discover the nature of growth of similar figures: linear measurements grow by constant factors, area by square numbers, and volume by cube numbers.

Name ______________________________ Date ______________

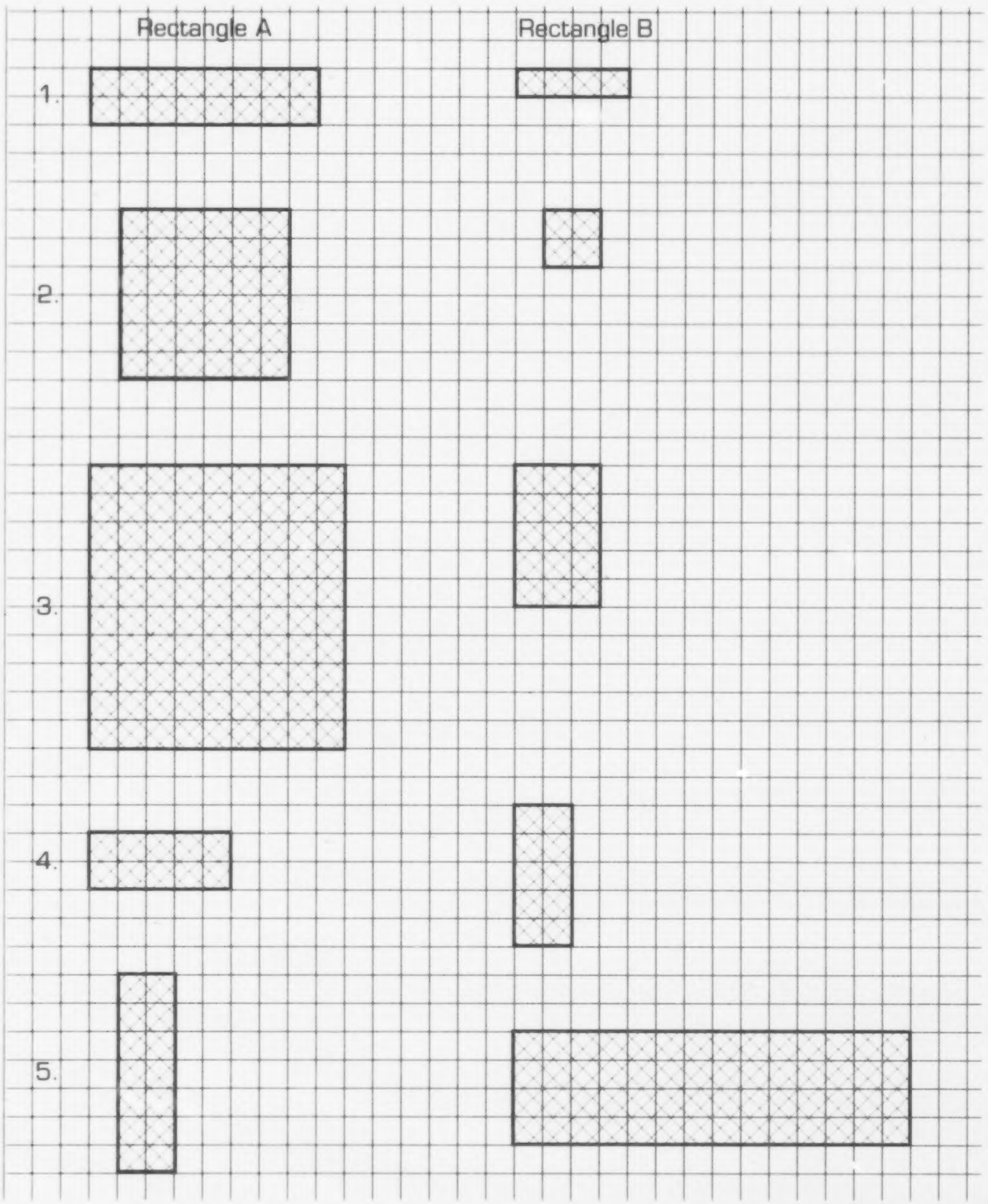
## PATTERN BLOCK MODEL OF CARLOS



**ACTIVITY SHEET 6.1**

82

Name ______________________________ Date ______________


Rectangle A
Rectangle B
1.
2.
3.
4.
5.

**ACTIVITY SHEET 6.2**

Name ______________________________ Date ______________

1. Using the rectangles on the first page of Activity Sheet 6.2, complete the table below. Find the length (L), width (W), perimeter (P), and area (A) of each rectangle.

2. What is a *ratio*? Give an example of a ratio.

3. For each rectangle express the ratio of its length to its width (*L* / *W*) and write it in the space provided in the table. Examine these ratios and describe any relationships you notice.

|  | Rectangle A |  |  |  |  | Rectangle B |  |  |  |  |
|---|---|---|---|---|---|---|---|---|---|---|
|  | L | W | P | A | L/W | L | W | P | A | L/W |
| 1. |  |  |  |  |  |  |  |  |  |  |
| 2. |  |  |  |  |  |  |  |  |  |  |
| 3. |  |  |  |  |  |  |  |  |  |  |
| 4. |  |  |  |  |  |  |  |  |  |  |
| 5. |  |  |  |  |  |  |  |  |  |  |

4. Using the data from the table above, express the ratios comparing lengths, widths, perimeters, and areas in the following table. Examine these ratios and describe any relationships you notice.

|  | Length of A to Length of B | Width of A to Width of B | Perimeter of A to Perimeter of B | Area of A to Area of B |
|---|---|---|---|---|
| 1. |  |  |  |  |
| 2. |  |  |  |  |
| 3. |  |  |  |  |
| 4. |  |  |  |  |
| 5. |  |  |  |  |

5. What are *similar figures?* Which of the rectangles on Activity Sheet 6.2 are similar? Explain how you know.

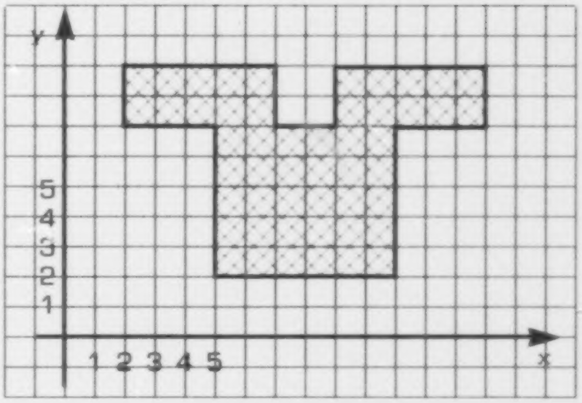
6. If the ratio of the lengths of two similar rectangles is 5:1, what will be the ratios of the areas? Explain.

7. If the ratio of the areas of two similar rectangles is 49:16, what is the ratio of the widths? Explain.

***ACTIVITY SHEET 6.2 (CONTINUED)***

Name ______________________________ Date _______________

*Directions:* Perform the following operations on the coordinates of the pattern for Carlos's shirt.



1. Multiply all the $x$-coordinates by 2 while leaving the $y$-coordinates unchanged. Plot the points and connect them.

2. Multiply all the $y$-coordinates by 2 while leaving the $x$-coordinates unchanged. Plot the points and connect them.

3. Multiply both coordinates by 2. Plot the points and connect them.

Organize your results in the chart below.

| Carlos's Shirt $(x, y)$ | Shirt 1 $(2x, y)$ | Shirt 2 $(x, 2y)$ | Shirt 3 $(2x, 2y)$ |
|---|---|---|---|
| (7, 9) | | | |
| (2, 9) | | | |
| (2, 7) | | | |
| (5, 7) | | | |
| (5, 2) | | | |
| (11, 2) | | | |
| (11, 7) | | | |
| (14, 7) | | | |
| (14, 9) | | | |
| (9, 9) | | | |
| (9, 7) | | | |
| (7, 7) | | | |

4. Are any shirts similar to Carlos's? If so, which ones? Explain how you know.

**ACTIVITY SHEET 6.3**

Name ______________________________ Date ______________

1. In each example, verify that the three rectangular solids A, B, and C are similar. Calculate the volume of each solid and determine the ratios of the linear measurements and of the volumes, as indicated.

| | Rectangular Solid A | | | | Rectangular Solid B | | | | Rectangular Solid C | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Length | Width | Height | Volume | Length | Width | Height | Volume | Length | Width | Height | Volume |
| 1. | 1 | 1 | 1 | | 2 | 2 | 2 | | 3 | 3 | 3 | |
| 2. | 1 | 2 | 2 | | 2 | 4 | 4 | | 3 | 6 | 6 | |
| 3. | 1 | 2 | 3 | | 2 | 4 | 6 | | 3 | 6 | 9 | |
| 4. | 2 | 3 | 5 | | 2 | 6 | 10 | | 6 | 9 | 15 | |

| | Ratio of Linear Measurements of | Ratio of Volumes |
|---|---|---|
| | Solid B to Solid A | Volume B/Volume A |
| 1. | | |
| 2. | | |
| 3. | | |
| 4. | | |

| | Solid C to Solid A | Volume C/Volume A |
|---|---|---|
| 1. | | |
| 2. | | |
| 3. | | |
| 4. | | |

| | Solid C to Solid B | Volume C/Volume B |
|---|---|---|
| 1. | | |
| 2. | | |
| 3. | | |
| 4. | | |

2. If the ratio of the lengths of two similar rectangular solids is *a/b*, what will be the ratio of the volumes?

**ACTIVITY SHEET 6.4**

## INVESTIGATION 7: WHAT IN THE WORLD ARE FRACTALS?

*Tami Martin*

In this investigation, students apply rational number and geometric concepts as they explore the topic of dimension. Most students have seen objects that can be used to represent various dimensions. For example, the tip of a needle, a piece of thread, a piece of paper, and a block can model zero-, one-, two-, and three-dimensional entities, respectively. But can an object exist that has a dimension between 2 and 3? As students examine this question, they use fractional exponents and ratios to learn about objects of fractional dimension, or *fractals*.

***Fractal-like objects can be found in nature. A fern is the classic example, in which each small piece resembles the entire plant. The close relationship between fractals and nature is evident in the surprising fact that mathematicians can now use computer-generated fractals to create forged images of many natural objects. Excellent pictures of forged ferns and mountains can be found in Peterson's The Mathematical Tourist (1988).***

*Prior knowledge:* Whole-number exponents and notation; $y^x$ key on the calculator; the concept of dimension; ratio; scale; similarity

*Concepts:* Dimensions, fractional dimensions, and fractals; self-similarity; size ratio; scale factor

*Objectives:*
1. To discuss the dimensions of various objects
2. To create self-similar shapes for figures of various dimensions
3. To analyze the ratios of new length to old length (the scale factor) and new size to old size (the size ratio) as self-similar figures are created
4. To analyze the relationship between scale factor and size ratio
5. To create fractals and analyze fractal data
6. To use fractional exponents in expressing the dimension of fractals

### Activity 1: What Is Dimension?

*Materials:* One copy of Activity Sheet 7.1 for each student; overhead projector

*Procedure:* 1. Discuss the meaning of the term *dimension*. Dimension is a fairly abstract term and not easily defined. The basic idea is that an object's dimension is a measure of how much space it takes up. The best way to develop an understanding of the concept of dimension at the middle school level is through a discussion of examples. A block is a three-dimensional object because it takes up room in three perpendicular directions (directions that are perpendicular to one another); the block is long, broad, and thick. A square is a two-dimensional object because it takes up room in only two perpendicular directions; the square is long and broad, but it has no thickness. A line is one-dimensional because it takes up room in only one direction; the line is long, but it has no breadth or thickness. A point is zero-dimensional because it takes up no space at all; the point has no length, breadth, or thickness.

***Edwin Abbott's classic, Flatland, tells the story of A Square's life in a two-dimensional world. A Square visits the inhabitants of a one-dimensional world and receives a visit from a sphere in his own two-dimensional world. Although the satirical view of Victorian life may not be fully appreciated, students may enjoy reading this delightful tale as a further exploration into the concept of dimension. As a follow-up creative writing assignment, students can write stories about what life might be like on a fractal.***

Every object we can touch is necessarily three dimensional. However, we can use certain three-dimensional objects to *represent* objects in lower dimensions. For example, a piece of thread, which has a very small breadth and is very thin, is a good model of a one-dimensional object, because we can imagine that length is its only measurable attribute. (See point 5 for a discussion of the role of mathematical models.) Ask students for examples of objects, especially real-world, everyday objects, of each dimension.

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

2. Distribute copies of Activity Sheet 7.1. Have students read the opening paragraph and enter the names of objects in the appropriate columns.

3. Have students work in pairs to fill in other examples of both mathematical objects (e.g., point, line, square, and cube) and everyday objects (e.g., table top, shoebox, classroom, and pencil point) on their sheets. Have students describe in their own words the critical attributes of an object from each dimension (e.g., a two-dimensional object may look flat from a local perspective) and write these descriptions at the bottom of the activity sheet.

***Students may characterize a two-dimensional object as being flat, but this is only true on a local level, not necessarily on a global level. For example, the surface of a sphere is two dimensional but certainly not flat (although a very small region would look flat). A better characterization would be "long and wide."***

4. After a few minutes, ask students to share some examples with the class. Record these on an overhead transparency. Ask students to share their descriptions of the essential attributes of some of the objects.

5. Use an everyday, two-dimensional object that students suggested to discuss this object's role as a mathematical model. For example, ask students if a piece of paper has length and width (*Of course*). Does it have height? If students doubt that a piece of paper has a thickness, ask them about ten pieces of paper. What would they observe about a stack of 100 sheets of paper? They should see that although paper has a *very small* height, it still has one. This demonstrates that although a piece of paper is very much like a two-dimensional object, it is really three dimensional. Explain to students that paper is a good example of a mathematical model—it is a real object we can see and touch that reminds us of an abstract mathematical object that is harder for us to imagine. Since all everyday objects are necessarily three dimensional, have students go through their lists and identify, by marking with a star, those everyday objects they have listed that are really just mathematical models. Below is an example of a completed chart:

***Mathematics as Communication: Students need to discuss the notion of "models" as they are used to represent meaning in mathematics. The use of models helps learners to visualize and concretize abstract concepts. Providing many examples and building on students' informal experiences may help clarify ideas.***

**Objects of Each Dimension**

| Zero | One | Two | Three |
|---|---|---|---|
| point | line | plane | space |
| *pinhead | curve | square | cube |
| *pencil point | *string | triangle | prism |
| | *thread | *piece of paper | sphere |
| | | *table top | brick |
| | | *chalkboard | shoebox |
| | | | classroom |

* Indicates mathematical model of given dimension

6. Ask students to identify some numbers between 2 and 3 (e.g., 2 1/4, 2.5, 2 7/8). Ask, "Can an object exist that has dimension between 2 and 3?" Take a "yes," "no," and "I have no idea" vote. Ask anyone who voted yes to suggest what such an object might look like. (This activity could make an interesting creative writing assignment.) This is a "thought question"—its answer will be explored in upcoming activities.

***"Each dimension ... corresponds to its own growth exponent. A surprising fact is that there are geometric objects whose growth exponents are not whole numbers. These strange objects, which have a kind of 'fractional dimension,' are examples of a fascinating collection of geometric patterns known as 'fractals'" (Banchoff 1990, p. 25).***

### *Activity 2: Exploration with Squares*

*Materials:* Activity Sheet 7.2; overhead projector; overhead transparency of Activity Sheet 7.2; at least twenty-five square blocks for each group of three to four students, such as color tiles or square blocks from a set of pattern blocks—ceramic tiles also work well.

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

*Procedure:* 1. Have students work in groups of three or four. Each student needs one copy of Activity Sheet 7.2 and a supply of square blocks. Remind students that these blocks will be used as mathematical models to represent squares. Let each side of a square have a length of one unit.

2. Ask students to find the fewest number of squares needed to make the next largest square (four squares are needed to make the next largest square, which is a $2 \times 2$ square). Why are these two shapes considered to be similar? (*Although the sizes are different, the shapes are the same, their corresponding angles are congruent, and their corresponding sides are in proportion.*) Allow students to discuss their ideas.

3. We need to define a new term: *size*. The *size* of a figure is calculated by counting the number of replicas (small pieces that make up the figure) that are needed to form the figure. In our problem, a replica is the original square block with edges of length 1. The original square is made up of one small square, so its size is 1.

***Role of the Teacher: In this example, the teacher gives and transmits information appropriately. Contrast this with student conjecturing, in which ground rules are given and students conjecture, test, and discuss their ideas without public evaluation by the teacher.***

Have students find the size of the new square (*4*) and the length of its sides (*2*). Enter this information in the table on Activity Sheet 7.2. Have students fill in "Squares" on the "DATA for" line.

Two more new terms are needed:

*Scale factor:* The *scale factor* between two similar figures (e.g., the small and large square) is the ratio of the new length to the old length.

$$\text{Scale factor} = \frac{\text{new length}}{\text{old length}}$$

*Size ratio:* The *size ratio* between similar figures is the ratio of the new size to the old size.

$$\text{Size ratio} = \frac{\text{new size}}{\text{old size}}$$

Have students find the scale factor and size ratio between the small $1 \times 1$ (old) and large $2 \times 2$ (new) squares. (*2/1 and 4/1, respectively*) Record the size ratio in the chart on Activity Sheet 7.2.

***The teacher allows students to wrestle with making sense of ideas, as they come to their own conclusions.***

4. Have students form larger squares and keep track of the data in Activity Sheet 7.2. They should always use the original $1 \times 1$ square as the old square. Although any two squares may be compared, scale factors and size ratios are fractional and may make the fundamental relationship harder to see. (See Extension.)

***Teachers need to be sensitive to students' use of mathematical vocabulary and language as they express themselves orally and in writing.***

5. Have students write a conjecture on Activity Sheet 7.2 about the relationship between the scale factor and the size ratio. Have them test the conjecture with a scale factor of 10. If not enough squares are available, have students combine groups or draw a model of the square-of-length-10 situation. (A conjecture might be "Multiply the scale factor by itself to get the size ratio for the square" or "Square the scale factor to get the size ratio for a square.")

6. Have students share their conjectures; write them on a transparency. Have the class verify that the conjectures are correct by substituting data into the equation or sentence. If the conjectures are stated in

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

prose, guide students in rephrasing the conjectures into equations. Encourage students to use exponents to express the relationship between the scale factor and the size ratio: (scale factor)$^2$ = size ratio.

***Mathematics as Reasoning: Verifying ideas***

*Extension:* As mentioned in point 4, students should use the original 1 × 1 square as the old square to more easily observe the relationship between the scale factor and the size ratio. An appropriate extension would be to have students examine the ratios between any two squares to verify that the relationships are the same. For example, to compare a "new" square with a length of 3 with an "old" square with a length of 2, $(3/2)^2 = 9/4$.

***Mathematics as Communication: Translating from words to equations***

***Activity 3: Exploration with Triangles***

*Materials:* Activity Sheet 7.2; unmarked overhead transparency of Activity Sheet 7.2; equilateral triangles, such as those in a set of pattern blocks, at least twenty-five for each group of three to four students

*Procedure:* 1. This activity is a repeat of Activity 2, using equilateral triangles instead of squares. Ask students to make a conjecture about how the scale factors and size ratios of larger triangles built from small triangles will compare with the findings for squares in Activity 2.

***Mathematics as Reasoning: Making a conjecture on the basis of prior experience***

2. Continue as in Activity 2. Remind students that the triangular blocks will be used as mathematical models to represent triangles. Each side of a triangle has a length of one unit.

3. Ask students to find the fewest number of triangles needed to make the next largest triangle. (*4*) Have students find the scale factor and size ratio between the small (old) and large (new) triangles, again using the original triangle as the old triangle. (*2/1 and 4/1, respectively*) Enter this information in the table on Activity Sheet 7.2. Ask students to continue building larger triangles and computing scale factors and size ratios until they feel confident about making a conjecture for triangles. Have students fill in "Triangles" on the "DATA for" line.

4. Compare the findings of this activity with Activity 2 (squares). Are the students surprised that the relationship is the same as that for the squares? Discuss why this may have occurred. Encourage students to use exponents to express the relationship between the scale factor and the size ratio: (scale factor)$^2$ = size ratio.

***Activity 4: Exploration with Cubes***

*Materials:* Activity Sheet 7.2; overhead transparency of Activity Sheet 7.2; cubes, such as Multilink cubes, 2-cm cubes, or any same-size cubes, at least sixty-four for each group of three to four students

*Procedure:* 1. This activity is a repeat of Activities 2 and 3, using cubes, which are three dimensional, instead of squares or triangles, which are two dimensional. Ask students to make a conjecture about how the scale factors and size ratios of larger cubes built from small cubes will compare with the findings for squares and triangles in Activities 2 and 3. Will they again be the same?

***Mathematics as Reasoning: This activity provides opportunities for exploring, conjecturing, testing, confirming, reformulating, or withdrawing a conjecture.***

2. Continue as in Activity 2. The length of each edge of the original cube is one unit. Ask students to find the fewest number of cubes needed to make the next larger cube. (*8*) Have students find the scale factor and size ratio between the small (old) and large (new) cubes, again using the original cube as the old cube. (*2/1 and 8/1, respectively*) Enter this information in the table on Activity Sheet 7.2. Have students fill in "Cubes" on the "DATA for" line.

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

***In two-dimensional figures, size ratio compares area. In three-dimensional figures, size ratio compares volume.***

As they work in groups, ask the students to continue to build larger cubes and collect data until they feel confident about making a conjecture for cubes.

3. Compare the findings of this activity with Activities 2 and 3 (squares and triangles). Are the students surprised that the relationship is different from that of the squares and triangles? Discuss why this may have occurred. Encourage students to use exponents to express the relationship between the scale factor and the size ratio: (scale factor)$^3$ = size ratio.

***Activity 5: Taking a "New" Look at Dimension***

***This activity could be done at the end of Activity 4 if there is time. It serves as an excellent closure to Activities 2 through 4.***

*Procedure:* 1. Ask students to discuss these questions: How were ratios used in Activities 2 through 4? Which measures were being compared? What information did the ratios give us?

2. Ask students to recall their conjectures about the relationship between the scale factor and size ratio for squares, triangles, and cubes. They should have arrived at the following conjectures:

For squares and triangles: (scale factor)$^2$ = size ratio

For cubes: (scale factor)$^3$ = size ratio

Ask students to recall the dimensions of squares, triangles, and cubes. What is the relationship between scale factor, size ratio, and dimension? This relationship is one way of defining dimension. Have students write a conjecture about the relationship among scale factor, size ratio, and dimension. Examples of students' conjectures are given under "Assessment."

Post this relationship in the room:

$$(\text{scale factor})^{\text{dimension}} = \text{size ratio}$$

***Assessment***

As a guide for making instructional decisions, the teacher should collect and read individual students' conjectures to learn more about students' understanding of the fundamental relationships. The following are some eighth-grade students' conjectures:

(1) Ken's and Linda's conjecture: For triangles,

(scale factor)$^2$ = size ratio.

(2) Christine's and Kevin's conjecture: The cube is three dimensional, so

(scale factor)$^3$ = size ratio

and $SF^{\text{dimension}} = SR$.

(3) Ryan's conjecture: I think we'd be using the fourth power next—in the fourth dimension.

*Comments about the students' conjectures:* The three conjectures reflect different levels of interpretation. In (1), Ken and Linda are focusing on one example. In (2), Christine and Kevin are using the cube example (and perhaps the previous two-dimensional examples) to make a generalization. Ryan is projecting ahead. For instruction, Ken and Linda need to be guided to examine all the examples and try to find a pattern. Christine, Kevin, and Ryan are ready to move on to the next activity.

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

***Diagnostic Assessment***

Before presenting Activity 6, assess students' understanding of exponentiation and their facility using a calculator. For example, consider including some of the following examples:

(1) $2^x = 8$ (2) $2^y = 16$ (3) $2^z = 12$

Think about this: Since 12 is halfway between 8 and 16, should $z$ be halfway between $x$ and $y$?

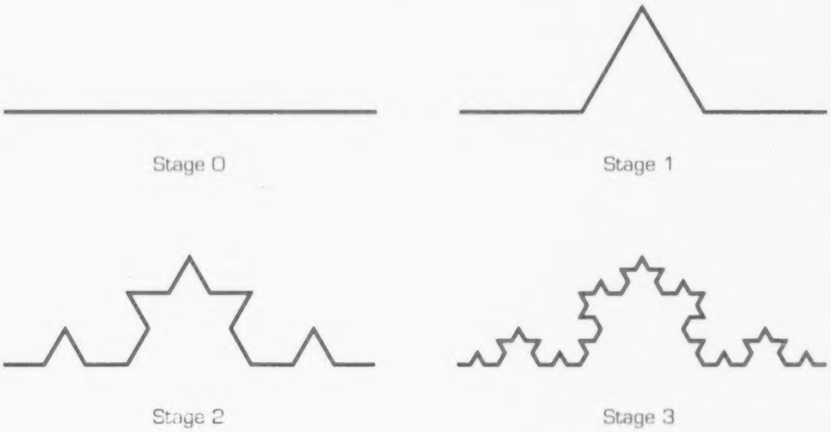
***Activity 6: Exploring Fractals***

*Materials:* Activity Sheet 7.2; Overhead Master 7.1; calculators with $y^x$ function (one for each student)

*Procedure:* 1. Ask students to recall the definition of dimension they developed from their experimentation with two- and three-dimensional objects ($(\text{scale factor})^{\text{dimension}} = \text{size ratio}$).

***The definition is recalled, but later, students can formulate their own iterative rules to analyze.***

Starting with a one-dimensional line, create a figure with a different dimension by adding on small pieces in another direction. The figure is formed by repeating (iterating) the following rule, which could be written on the chalkboard or on an overhead transparency: "Replace the middle third of every line segment with an upside down 'V,' each of whose two sides is equal in length to the segment removed." A line of length 1 is called stage 0. Draw the first three stages in the development of this figure on the chalkboard or an overhead transparency. Ask students to try drawing the fourth stage. Finally, draw the fourth stage, restating the iteration rule and pointing out how this rule has been applied at each stage. See figure 7.1.


Stage 0

Stage 1

Stage 2

Stage 3

***Fig. 7.1. Stages in the development of a fractal***

2. Have students compute the fractal dimension of this figure by following the four steps on Overhead Master 7.1. Work through the steps with the class, filling in the blanks as you go. In figure 7.1, step 1 shows stage 1 and an enlarged stage 2. In steps 2 and 3, students compute the scale factor and size ratio by referring to the figures.

The distinction between length and size can be tricky for students. To help minimize confusion, emphasize that length is measured in a straight line along the bottom of the figure. Here stage 1 is the "original" and has

***Making the distinction between length and size can be difficult. Remind students that length is determined by measuring a straight line, but size is found by counting "whole copies" of the original figure.***

length 1. Thus, the enlarged stage 2 would measure three units in a straight line across the bottom. Students can highlight the size by putting a circle around the four replicas of stage 1 appearing in the enlarged stage 2.

In step 4, the students apply the definition of dimension. Have them use the calculator as a tool to estimate the dimension of a figure. Since the scale factor is 3 and the size ratio is 4, we have $3^d = 4$ (when $d$ represents dimension). Ask students how they could use their calculators to estimate $d$. Write on an overhead transparency: $3^1 = 3$ and $3^2 = 9$. Allow students to discuss that since $3^1$ is 3 (which is less than 4) and $3^2$ is 9 (which is greater than 4), the dimension of this figure must be greater than 1 but less than 2. Ask students to estimate another value for $d$, say 1.5, and have them test it with their calculators. (Calculator code: $\boxed{3}\ \boxed{y^x}\ \boxed{1}\ \boxed{.}\ \boxed{5}\ \boxed{=}$) Write $3^{(1.5)} = 5.196$. Since 5.196 is still greater than 4, students should adjust their estimate of the dimension. Have them continue experimenting and adjusting their estimates until they think they are "close enough." Two or three decimal places of accuracy should be enough. For this fractal, $d = 1.262$. Write the final equation on the transparency.

***Technology: Using the calculator as a tool***

3. Remind students that the figure they were just working on, of dimension greater than 1 but less than 2, was derived by iteratively adding on to a one-dimensional line segment. Explain that objects like these, with fractional dimension, are known as *fractals.* Fractals are infinitely self-similar objects (with each small piece resembling the entire figure) formed by repeated additions to, or removals from, a figure. The object obtained at the limit of the repeated procedure is the fractal.

***The notion of limit is important in mathematics. Although it is hard to imagine what happens to the figure after an infinite number of iterations, students may try to imagine the fractal at stage 100.***

***Activity 7: Another Fractal***

*Materials:* Overhead Master 7.2, calculators with $y^x$ function (one for each student)

*Procedure:* 1. Have students use the four-step process from the previous example to determine the dimension of the figure obtained by repeating the following iteration rule, which could be written on the chalkboard or on a transparency: "Remove the middle quarter of each triangle." Construct stages 0, 1, and 2 with the students. The stages can be displayed using Overhead Master 7.2 (showing only the triangle iterations). Have students draw and discuss stage 3.

2. Ask students to write a conjecture about the dimension of the fractal before actually computing it with the calculator. Between which two whole numbers will the dimension lie? To which one will it be closer?

3. Have students complete the four-step procedure to estimate the dimension of the fractal. Again, let them determine a reasonable level of accuracy for the decimal approximation. In this example, $2^d = 3$, so $d = 1.585$.

*Extension:* Have students create and analyze the dimension described as follows: "Add a cube to the center ninth of each square region." The square iteration on the bottom of Overhead Master 7.2 can be used to illustrate stages 0, 1, and 2. In this example, $3^d = 13$, so $d = 2.335$.

***Activity 8: Organizing Fractal Data***

*Materials:* Activity Sheet 7.3; an overhead transparency of Activity Sheet 7.3; overhead projector

*Procedure:* 1. Have students examine the three fractals whose dimen-

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

sion they have computed (the line from Activity 6; the triangle and the square from Activity 7 and the extension). Have students fill in the chart on Activity Sheet 7.3 for each fractal (the dimension of starting figure, the iteration rule, and the dimension of the fractal for each of the three fractals they have created).

When complete, the chart should look something like the one below:

| Dimension of Figure at Stage 0 | Iteration Rule | Dimension of the Fractal |
|---|---|---|
| Line (1 dimension) | Add upside-down V over middle third. | 1.262 |
| Triangle (2 dimension) | Remove middle quarter triangle. | 1.585 |
| Square (2 dimension) | Add cube over middle ninth of every square. | 2.335 |

Summarize these results on an overhead transparency of Activity Sheet 7.3.

Ask students to notice that fractals of dimension greater than 1 but less than 2 were formed by adding to a one-dimensional figure or removing from a two-dimensional figure. Have them suggest ways to make fractals of dimension between 2 and 3, greater than 3, or less than 1.

### *Performance Assessment*

Have student groups create original fractals by defining starting figures and iterative rules. Have the groups draw the first few stages of their fractal to display in the classroom. Have groups compute the dimensions of one another's fractals and compare results. As an extension, have students create fractals of dimension 1/2 or 1/3. (These can be formed by iteratively removing portions of a straight line segment.) To grade such an open-ended assignment, use a four- or six-point holistic scoring scale (Stenmark 1991).

*Extension 1:* If the coast of North America were a straight line, it would have dimension 1. However, the many jagged edges caused by rocks, sand, water, and dunes increase the dimension to a rational number greater than 1. Ask students to make a conjecture about the two integral values between which the fractal dimension of a coastline lies. (It should be more than 1 but less than 2.)

*Extension 2:* Ask the original question, "Can an object exist that has dimension between 2 and 3?" They have discovered that the answer is yes. Discuss the surprising nature of the answer to this question. It seems odd that a dimension between 2 and 3 could exist, but objects in nature with jagged edges, like the coast or a fern, may be thought of as having fractional dimension.

*Extension 3:* Brainstorm with the class about other self-similar fractal-like objects that they may know of. Have students write a justification of why they believe the object has a fractional dimension and between which two whole-number dimensions the objects lie. If possible, show slides or pictures of computer-generated fractal forgeries (e.g., trees, mountains, valleys, and clouds). For a collection of fractal slides, see Peitgen et al. (1991).

***"The concept of fractal is relatively new yet is quickly being recognized as a realistic way to model nature" (Kern and Mauk 1990, p. 179).***

*Extension 4:* Students can create their own fractals by writing programs in Pascal (Camp 1991) or Logo (Kern and Mauk 1990). For further investigations, see Peitgen et al. (1991).

> ***"Since the creation of fractals usually requires a process that is applied an infinite number of times, it is only with the advent of modern computer graphics that it has been possible to carry out the experiments necessary to explore them effectively" (Banchoff 1990, p. 25).***

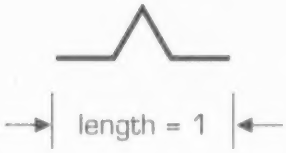
### ***CLOSING COMMENTS***

Rational numbers and proportions are valuable tools for problem solving both in school and out of school. To help middle grades students see the connection between these two mathematical settings, they need to encounter rational number and proportion activities that build on their informal knowledge and language. Some informal, everyday tasks, which include baking, buying, building, fixing, and playing, provide a foundation on which more formal rational number and proportion concepts may develop.

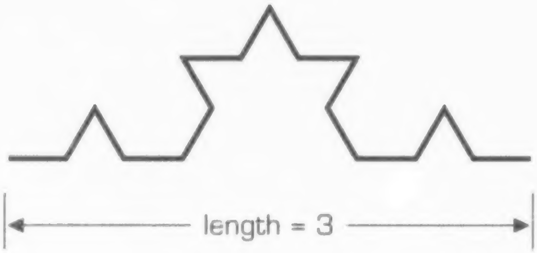
In support of the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989), this collection of investigations builds on students' knowledge and interests and provides some examples of how rational numbers serve to describe and measure occurrences in imaginary and real worlds. As students experience and appreciate the power and nature of rational numbers and proportions by applying them in meaningful contexts, they will realize that these concepts are not as onerous as they may appear. Middle grades students' facility with rational numbers and proportions will provide a firm foundation for more advanced study in mathematics.

# CALCULATING THE DIMENSION OF A FRACTAL

1. Start with any stage. Draw an enlarged version of the next stage so that it contains exact (same size) copies of the previous stage.



Stage 1



Stage 2

2. Find the scale factor.

$$\text{scale factor} = \frac{\text{new length}}{\text{old length}} = \underline{\qquad} =$$

3. Find the size ratio.

$$\text{size ratio} = \frac{\text{new size}}{\text{old size}} = \underline{\qquad} =$$

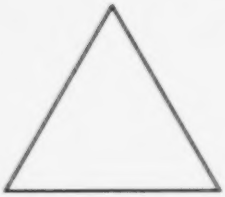
4. Use the definition of dimension to estimate dimension (*d*).

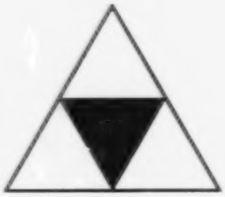
$$(\text{scale factor})^d = \text{size ratio}$$

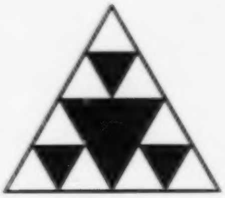
**OVERHEAD MASTER 7.1**

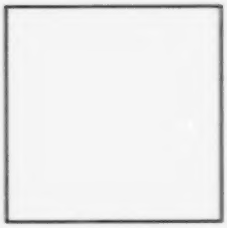
# MORE FRACTALS

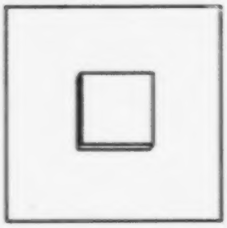
Objects with fractional dimension are known as *fractals*. Fractals are infinitely self-similar objects (with each small piece resembling the entire figure) formed by repeated additions to, or removals from, a figure. The object obtained at the limit of the repeated procedure is the fractal.

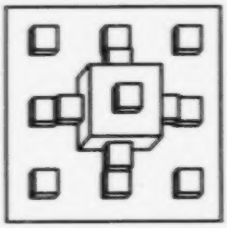

Stage 0


Stage 1


Stage 2


Stage 0


Stage 1


Stage 2

**OVERHEAD MASTER 7.2**

Name ______________________ Date ______________

## OBJECTS OF VARIOUS DIMENSIONS

Most of the mathematical objects you have studied have dimensions that are whole numbers. For example, such solids as cubes and icosahedrons have dimension 3. Squares, triangles, and many other planar figures are two dimensional. Lines are one dimensional, and points have dimension 0.

List some mathematical and everyday objects of each dimension.

| ZERO | ONE | TWO | THREE |
|---|---|---|---|
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |
| | | | |

Describe some of the features of the objects listed in the chart above.

**ACTIVITY SHEET 7.1**

Name ______________________ Date ______________

Recall the following definitions:

$$\text{Scale factor} = \frac{\text{new length}}{\text{old length}}$$

$$\text{Size ratio} = \frac{\text{new size}}{\text{old size}}$$

Record the data for various scale factors. (Suggestion: It's easier to always use the original object of size 1 for the "old length" and "old size" when finding the scale factor and the size ratio.)

DATA for ______________

| Length | Size | Scale Factor | Size Ratio |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | ______ | 2 | ______ |
| 3 | ______ | ______ | ______ |
| 4 | ______ | ______ | ______ |
| 5 | ______ | ______ | ______ |
| 6 | ______ | ______ | ______ |
| 7 | ______ | ______ | ______ |
| 8 | ______ | ______ | ______ |
| 9 | ______ | ______ | ______ |
| 10 | ______ | ______ | ______ |

Write a conjecture about the relationship between the scale factor and the size ratio:

**ACTIVITY SHEET 7.2**

Name ______________________________ Date ________________

| Dimension of Figure at Stage 0 | Iteration Rule | Dimension of the Fractal |
|---|---|---|
| | | |
| | | |
| | | |
| | | |
| | | |
| | | |
| | | |
| | | |
| | | |
| | | |
| | | |
| | | |
| | | |
| | | |
| | | |

***ACTIVITY SHEET 7.3***

## SELECTED BIBLIOGRAPHY

References and suggested resources on recycling, similarity, and fractals can be found after "General Resources including Fractions, Ratios, and Proportions" under their respective headings.

### General Resources including Fractions, Ratios, and Proportions

Abbott, Edwin. *Flatland.* New York: Dover, 1952.

Apostol, Tom M. *Similarity.* Pasadena, Calif.: California Institute of Technology, Project Mathematics! 1990. Videotape. (Distributed by NCTM)

Artzt, Alice F. "The Secondary Mathematics Curriculum in Action: Using the Copy Machine to Explore Ideas of Congruence and Similarity." *New York State Mathematics Teachers Journal* 41 (1): 22–24.

Artzt, Alice F., and Claire M. Newman. *How to Use Cooperative Learning in the Mathematics Class.* Reston, Va.: National Council of Teachers of Mathematics, 1990.

Banchoff, Thomas F. "Dimension." In *On the Shoulders of Giants: New Approaches to Numeracy*, edited by Lynn A. Steen, pp. 11–59. Washington, D.C.: National Academy Press, 1990.

Baroody, Arthur J. "One Point of View: Manipulatives Don't Come with Guarantees." *Arithmetic Teacher* 37 (October 1989): 4–5.

Bennett, A. B., Jr., and Patricia S. Davidson. *Fraction Bars.* Fort Collins, Colo.: Scott Resources, Inc., 1973.

Behr, Merlyn, Richard Lesh, Thomas R. Post, and Edward A. Silver. "Rational-Number Concepts." In *Acquisition of Mathematics Concepts and Processes*, edited by Richard Lesh and Marsha Landau, pp. 91–126. Orlando, Fla.: Academic Press, 1983.

Behr, Merlyn, and Thomas R. Post. "Teaching Rational Number and Decimal Concepts." In *Teaching Mathematics in Grades K–8: Research-based Methods*, 2nd ed., edited by Thomas R. Post, pp. 201–48. Boston: Allyn & Bacon, 1992.

Berman, Barbara, and Fredda Friederwitzer. *Fraction Circle Activities: A Sourcebook for Grades 4–8.* Palo Alto, Calif.: Dale Seymour Publications, 1988.

Bezuk, Nadine S. "Fractions—Easy Pieces." *Arithmetic Teacher* 36 (February 1989): 3.

Bezuk, Nadine S., and Barbara E. Armstrong. "Understanding Fraction Multiplication." *Mathematics Teacher* 85 (December 1992): 729–33, 739–44.

———. "Understanding Division of Fractions." *Mathematics Teacher* 86 (January 1993):43–46, 56–60.

Bezuk, Nadine S., and Marilyn Bieck. "Current Research on Rational Numbers and Common Fractions: Summary and Implications for Teachers." In *Research Ideas for the Classrooom: Middle Grades Mathematics*, edited by Douglas T. Owens, pp. 118–36. New York: Macmillan, 1993.

Bezuk, Nadine S., and Kathleen Cramer. "Teaching about Fractions: What, When, and How?" In *New Directions for Elementary School Mathematics*, 1989 Yearbook of the National Council of Teachers of Mathematics, edited by Paul R. Trafton, pp. 156–67. Reston, Va.: The Council, 1989.

Bright, George W., and John G. Harvey. "Using Games to Teach Fraction Concepts and Skills." In *Mathematics for the Middle Grades (5–9)*, 1982 Yearbook of the National Council of Teachers of Mathematics, edited by Linda Silvey, pp. 205–16. Reston, Va.: The Council, 1982.

Burns, Marilyn. *A Collection of Math Lessons from Grades 3 through 6.* Sausalito, Calif.: Math Solutions Publications, 1987.

———. *The Math Solution: Teaching for Mastery through Problem Solving.* Sausalito, Calif.: Marilyn Burns Education Associates, 1984.

Camp, Dane R. "A Fractal Excursion." *Mathematics Teacher* 81 (April 1991): 265–75.

Carnegie Council on Adolescent Development. *Turning Points: Preparing American Youth for the Twenty-first Century.* The Report of the Task Force on Education of Young Adolescents. Washington, D.C.: The Council, 1989.

Carpenter, Thomas P., Elizabeth Fennema, and Thomas A. Romberg, eds. *Rational Numbers: An Integration of Research.* Hillsdale, N.J.: Erlbaum, 1993.

Corwin, Rebecca B., and Susan Friel. "Statistics: Sampling" [Pullout Section]. *Elementary Mathematician*, n.d.

Corwin, Rebecca B., Susan Jo Russell, and Cornelia C. Tierney. *Seeing Fractions*. Sacramento: California Department of Education, 1990.

Cramer, Kathleen, and Nadine Bezuk. "Multiplication of Fractions: Teaching for Understanding." *Arithmetic Teacher* 39 (November 1991): 34–37.

Cramer, Kathleen, Nadine Bezuk, and Merlyn Behr. "Proportional Relationships and Unit Rates." *Mathematics Teacher* 82 (October 1989): 537–44.

Cramer, Kathleen, Thomas Post, and Sarah Currier. "Learning and Teaching Ratio and Proportion: Research Implications." In *Research Ideas for the Classroom: Middle Grades Mathematics*, edited by Douglas T. Owens, pp. 159–78. New York: Macmillan, 1993.

Crimmons, James C. (Executive Producer). *The Challenge of the Unknown*. Videotape. Bartlesville, Okla., 1986. (Available free from Karol Media, P.O. Box 7600, Wilkes-Barre, PA 18773.)

Curcio, Frances R. *Developing Graph Comprehension: Elementary and Middle School Activities*. Reston, Va.: National Council of Teachers of Mathematics, 1989.

———. "Mathematics as Communication: Using a Language-Experience Approach in the Elementary Grades." In *Teaching and Learning Mathematics in the 1990s*, 1990 Yearbook of the National Council of Teachers of Mathematics, edited by Thomas J. Cooney, pp. 69–75. Reston, Va.: The Council, 1990.

Curcio, Frances R., Francine Sicklick, and Susan B. Turkel. "Divide and Conquer: Unit Strips to the Rescue." *Arithmetic Teacher* 35 (December 1987): 6–12.

Davidson, Neil, ed. *Cooperative Learning in Mathematics*. Menlo Park, Calif.: Addison-Wesley Publishing Co., 1989.

Educational Development Center, Inc. *My Travels with Gulliver*. Scotts Valley, Calif.: Wings for Learning, 1991.

Fey, James T., William M. Fitzgerald, Susan N. Friel, Glenda Lappan, and Elizabeth Phillips. *Understanding Fractions, Decimals, and Percents*. The Connected Mathematics Project. East Lansing, Mich.: Michigan State University, in press.

Geddes, Dorothy. *Geometry in the Middle Grades*. Reston, Va.: National Council of Teachers of Mathematics, 1992.

———. *Measurement in the Middle Grades*. Reston, Va.: National Council of Teachers of Mathematics, 1994.

Gibb, E. Glenadine, Phillip S. Jones, and Charlotte W. Junge. "Number and Operation." In *The Growth of Mathematical Ideas, Grades K–12*, Twenty-fourth Yearbook of the National Council of Teachers of Mathematics, edited by Phillip S. Jones, pp. 7–64. Washington, D.C.: The Council, 1959.

Haldane, J. B. S. "On Being the Right Size." In *The World of Mathematics: Vol. 2*, edited by James R. Newman, pp. 952–57. New York: Simon and Schuster, 1956.

Hart, Kathleen. "Ratio and Proportion." In *Number Concepts and Operations in the Middle Grades*, edited by James Hiebert and Merlyn Behr, pp. 198–219. Hillsdale, N.J.: Erlbaum, and Reston, Va.: National Council of Teachers of Mathematics, 1988.

Hirsch, Christian R., and Glenda Lappan. "Transition to High School Mathematics." *Mathematics Teacher* 82 (November 1989): 614–18.

Hoffer, Alan R., and Shirley Ann K. Hoffer. "Ratios and Proportional Thinking." In *Teaching Mathematics in Grades K–8: Research-based Methods*, 2nd ed., edited by Thomas R. Post, pp. 303–30. Boston: Allyn & Bacon, 1992.

Kern, Jane F., and Cherry C. Mauk. "Exploring Fractals—a Problem-Solving Adventure Using Mathematics and Logo." *Mathematics Teacher* 83 (March 1990): 179–85, 244.

Kieran, Thomas E. "Knowing Rational Numbers: Ideas and Symbols." In *Selected Issues in Mathematics Education*, edited by Mary M. Lindquist, pp. 69–81. Chicago: National Society for the Study of Education, and Reston, Va.: National Council of Teachers of Mathematics, 1980.

———. "Personal Knowledge of Rational Numbers: Its Intuitive and Formal Development." In *Number Concepts and Operations in the Middle Grades*, edited by James Hiebert and Merlyn Behr, pp. 162–81. Hillsdale, N.J.: Erlbaum, and Reston, Va.: National Council of Teachers of Mathematics, 1988.

**BEST COPY AVAILABLE**

◆ ◆ ◆ ◆ ◆ ◆ ◆ ◆

———. "On the Mathematical, Cognitive, and Instructional Foundations of Rational Numbers." In *Number and Measurement*, edited by Richard A. Lesh, pp. 101–44. Columbus, Ohio: ERIC/Science, Mathematics, and Environmental Education Information Analysis Center, 1976.

Kieran, Thomas, and Merlyn Behr. "Fractions: Summary." In *Theory, Research, and Practice in Mathematical Education*, edited by Alan Bell, B. Low, and Jeremy Kilpatrick, pp. 176–85. Nottingham, England: Shell Centre for Mathematical Education, 1985.

Kouba, Vicky L., Catherine A. Brown, Thomas P. Carpenter, Mary M. Lindquist, Edward A. Silver, and Jane O. Swafford. "Results of the Fourth NAEP Assessment of Mathematics: Number, Operations, and Word Problems." *Arithmetic Teacher* 35 (April 1988): 14–19.

Kroll, Diana Lambdin, and Frank K. Lester, Jr. "Appendix 1: Evaluation." In *Developing Number Sense in the Middle Grades*, by Barbara Reys, pp. 50–54. Reston, Va.: National Council of Teachers of Mathematics, 1991.

Lappan, Glenda, William Fitzgerald, Mary Jean Winter, and Elizabeth Phillips. *Similarity and Equivalent Fractions*. Menlo Park, Calif.: Addison-Wesley Publishing Co., 1986.

Lay, L. Clark. "Mental Images and Arithmetical Symbols." *Visible Language* 16 (Summer 1982): 259–74.

Lesh, Richard, and Susan Lamon, eds. *Assessment of Authentic Performance in School Mathematics*. Washington, D.C.: American Association for the Advancement of Science, 1992.

*Litter Control, Waste Management, and Recycling Resource Unit, K–6*. Bulletin 1722. Baton Rouge, La.: Louisiana State Department of Education, 1985.

Mack, Nancy L. "Learning Fractions with Understanding: Building on Informal Knowledge." *Journal for Research in Mathematics Education* 21 (1990): 16–32.

Maddux, Hilary C., ed. *The Challenge of the Unknown Teaching Guide*. New York: W. W. Norton & Co., 1986.

National Council of Teachers of Mathematics. *An Agenda for Action*. Reston, Va.: The Council, 1980.

———. *Assessment in the Mathematics Classroom*. 1993 Yearbook of the National Council of Teachers of Mathematics, edited by Norman L. Webb. Reston, Va.: The Council, 1993.

———. *Curriculum and Evaluation Standards for School Mathematics*. Reston, Va.: The Council, 1989.

———. *Curriculum and Evaluation Standards for School Mathematics* Addenda Series, Grades K–6, edited by Miriam A. Leiva. Reston, Va.: The Council, 1991–93.

———. *Curriculum and Evaluation Standards for School Mathematics* Addenda Series, Grades 5–8, edited by Frances R. Curcio. Reston, Va.: The Council, 1991–94.

———. *Curriculum and Evaluation Standards for School Mathematics* Addenda Series, Grades 9–12, edited by Christian R. Hirsch. Reston, Va.: The Council, 1991–94.

———. *Professional Standards for Teaching Mathematics*. Reston, Va.: The Council, 1991.

Niess, Margaret L. "Mathematics and M&Ms." *Computing Teacher* 20 (August/September 1992): 29–31.

Ohlsson, Stellan. "Mathematical Meaning and Applicational Meaning in the Semantics of Fractions and Related Concepts." In *Number Concepts and Operations in the Middle Grades*, edited by James Hiebert and Merlyn Behr, pp. 53–92. Hillsdale, N.J.: Erlbaum, and Reston, Va.: National Council of Teachers of Mathematics, 1988.

Owens, Douglas T., and Douglas B. Super. "Teaching and Learning Decimal Fractions." In *Research Ideas for the Classroom: Middle Grades Mathematics*, edited by Douglas T. Owens, pp. 137–58. New York: Macmillan, 1993.

Peitgen, Heinz-Otto, Hartmut Jurgens, Dietmar Saupe, Evan Maletsky, Terry Perciante, and Lee Yunker. *Fractals for the Classroom: Strategic Activities*, Vol. 1. New York: Springer-Verlag, 1991.

Peterson, Ivars. *The Mathematical Tourist: Snapshots of Modern Mathematics*. New York: W. H. Freeman & Co., 1988.

Phillips, Elizabeth. *Patterns and Functions*. Reston, Va.: National Council of Teachers of Mathematics, 1991.

Post, Thomas R. "Fractions and Other Rational Numbers." *Arithmetic Teacher* 37 (September 1989): 3, 28.

Post, Thomas R., Merlyn J. Behr, and Richard Lesh. "Interpretations of Rational Number Concepts." In *Mathematics for the Middle Grades (5–9)*, 1982 Yearbook of the National Council of Teachers of Mathematics, edited by Linda Silvey, pp. 59–72. Reston, Va.: The Council, 1982.

Reys, Barbara J. *Developing Number Sense in the Middle Grades*. Reston, Va.: National Council of Teachers of Mathematics, 1991.

Roper, Ann, and Linda Holden Charles. *Fraction Circles Plus*. Sunnyvale, Calif.: Creative Publications, 1990.

Rowan, Thomas. "Report of the Task Force on Addenda to the NCTM K–12 *Curriculum and Evaluation Standards for School Mathematics*." Reston, Va.: National Council of Teachers of Mathematics. Unpublished report, November 1988.

*Rush to Burn*. *Newsday*. Washington, D.C.: Island Press, 1989, pp. 23–27.

Russell, Dorothy S., and Elaine M. Bologna. "Teaching Geometry with Tangrams." *Arithmetic Teacher* 30 (October 1982): 34–38.

Rybczynski, Witold. "We Are What We Throw Away." *New York Times Book Review*, 5 July 1992, pp. 5–6.

Schwartz, Richard H. *Mathematics and Global Survival*. Needham Heights, Mass.: Ginn Press, 1989.

Skemp, Richard. *Structured Activities for Primary Mathematics*. Vol. 2, *For the Later Years*. London: Routledge, 1989.

Steen, Lynn Arthur. "Teaching Mathematics for Tomorrow's World." *Educational Leadership* 47 (September 1989): 18–22.

Stenmark, Jean Kerr, ed. *Mathematics Assessment: Myths, Models, Good Questions, and Practical Suggestions*. Reston, Va.: National Council of Teachers of Mathematics, 1991.

Swetz, Frank, and J. S. Hartzler. *Mathematical Modeling in the Secondary School Curriculum*. Reston, Va.: National Council of Teachers of Mathematics, 1991.

Swift, Jonathan. *Gulliver's Travels*. Chicago: Kidsbooks, 1992.

Thompson, Patrick W. "The Development of the Concept of Speed and Its Relationship to Concepts of Rate." In *The Development of Multiplicative Reasoning in the Learning of Mathematics*, edited by G. Harel and Jere Confrey, pp. 181–236. Albany, N.Y.: SUNY Press, 1994.

"TNT News Blasts: Read or Digest." *3-2-1 Contact* (Children's Television Workshop), no. 128, July/August, 1992, p. 2.

Vergnaud, Gerard. "Multiplicative Structures." In *Acquisition of Mathematical Concepts and Processes*, edited by Richard Lesh and Marsha Landau, pp. 127–74. Orlando, Fla.: Academic Press, 1983.

———. "Multiplicative Structures." In *Number Concepts and Operations in the Middle Grades*, edited by James Hiebert and Merlyn Behr, pp. 141–61. Hillsdale, N.J.: Erlbaum, and Reston, Va.: National Council of Teachers of Mathematics, 1988.

Vygotsky, Lev. *Thought and Language*. Cambridge, Mass.: MIT Press, 1934/1986.

*Waste Away—Information and Activities for Investigating Trash Problems and Solutions for Upper Elementary and Junior High School Students*. Woodstock, Vt.: Vermont Institute of Natural Science, 1989.

Witherspoon, Mary Lou. "Fractions: In Search of Meaning." *Arithmetic Teacher* 40 (April 1993): 482–85.

Zawojewski, Judith. *Dealing with Data and Chance*. Reston, Va.: National Council of Teachers of Mathematics, 1991.

Zulie, Matthew E. *Fractions with Pattern Blocks*. Sunnyvale, Calif.: Creative Publications, 1975.

### ***Similarity***

Apostol, Tom M. *Similarity*. Pasadena, Calif.: California Institute of Technology, Project Mathematics! 1990. Videotape. (Distributed by NCTM)

BEST COPY AVAILABLE

Artzt, Alice F. "The Secondary Mathematics Curriculum in Action: Using the Copy Machine to Explore Ideas of Congruence and Similarity." *New York State Mathematics Teachers Journal* 41 (1): 22-24.

Billstein, Rick, and Jim Trudnowski. "Godzilla: Fact or Fiction?" *Student Math Notes* [National Council of Teachers of Mathematics], November 1989.

Educational Development Center, Inc. *My Travels with Gulliver*. Scotts Valley, Calif.: Wings for Learning, 1991.

Haldane, J. B. S. "On Being the Right Size." In *The World of Mathematics: Vol. 2*, edited by James R. Newman, pp. 952-57. New York: Simon and Schuster, 1956.

Lappan, Glenda, William Fitzgerald, Mary Jean Winter, and Elizabeth Phillips. *Similarity and Equivalent Fractions*. Middle Grades Mathematics Project. Menlo Park, Calif.: Addison-Wesley Publishing Co., 1986.

Mason, M. S. "Oh, You Beautiful Bug, You Great Big, Beautiful Bug." *Christian Science Monitor*, 4 August 1992, pp. 16-17.

Swift, Jonathan. *Gulliver's Travels*. Chicago: Kidsbooks, 1992.

**Recycling**

Arenstein, Walter Alan, and Ginny King Supple. "Solid Waste? It's Elementary: The Research and Development of Two Environmental Education Programs." *Journal of Environmental Education* 13 (Fall 1981): 3-11.

*Community Solutions for Solid Waste Pollution, Level 6—Teacher Guide: Operation Waste Watch*. Richmond, Va.: Virginia State Department of Waste Management, Division of Litter and Recycling, 1980.

DeLago, Louise Furia. "Recycling Research: Tracking Trash." *Science Teacher* 58 (April 1991): 32-35.

Grimes, William. "Seeking the Truth in Refuse." *New York Times*, 13 August 1992, pp. B1, B6.

Environment Canada. *Save It! A Practical Family Kit on Saving Resources, Saving Money, and ... Saving the Environment*. Edmonton, Alberta, Canada: Environment Canada Public Affairs, 1984. (ERIC Document Reproduction Service No. ED 263 002)

Heukerott, Pamela Beth. "Growing Garbage—Uncommon Lessons in Pollution." *Learning* 16 (October 1987): 33-34.

Howe, Robert W., and John F. Disinger. *Environmental Activities for Teaching Critical Thinking*. Columbus, Ohio: ERIC Clearinghouse for Science, Mathematics, and Environmental Education, 1990.

Indiana State Department of Education, Center for School Improvement and Performance. *Waste Reduction Guide for Indiana Schools*. Indianapolis, Ind.: Indiana State Department of Education, 1991. (ERIC Document Reproduction Service No. ED 338 489)

Linden, Eugene. "Is the Planet on the Back Burner?" *Time* 136 (24 December 1990): 48-50.

*Litter Control, Waste Management, and Recycling Resource Unit, K-6*. Bulletin 1722. Baton Rouge, La.: Louisiana State Department of Education, 1985.

Merline, J. W. "Trashing the Planet." *Consumer's Research Magazine* 74 (March 1991): 38.

O'Connor, Karen. *Garbage: Our Endangered Planet*. San Diego, Calif.: Lucent Books, 1989.

Parker, Tom. *In One Day*. Boston: Houghton Mifflin, 1984.

Rathje, William L. "Once and Future Landfills." *National Geographic* 179 (May 1991): 117-34.

*Rush to Burn*. *Newsday*. Washington, D.C.: Island Press, 1989, pp. 23-27.

Rybczynski, Witold. "We Are What We Throw Away." *New York Times Book Review*, 5 July 1992, pp. 5-6.

*Safe Planet: The Guide to Environmental Film and Video*. New York: Media Network, 1990. (ERIC Document Reproduction Service No. ED 331 710)

Scarlett, Lynn. "Will Recycling Help the Environment?" *Consumer's Research Magazine* 74 (March 1991): 17-18.

Schwartz, Richard H. *Mathematics and Global Survival.* Needham Heights, Mass.: Ginn Press, 1989.

Skidmore, Steve. *What a Load of Trash!* Brookfield, Conn.: Millbrook Press, 1990.

"TNT News Blasts: Read or Digest." *3-2-1 Contact* [Children's Television Workshop], no. 128, July/August, 1992, p. 2.

Virag, Irene. "The All-Consuming Lifestyle." In *Rush To Burn*, pp. 23–37. Washington, D.C.: Island Press, 1989.

*Waste Away—Information and Activities for Investigating Trash Problems and Solutions for Upper Elementary and Junior High School Students.* Woodstock, Vt.: Vermont Institute of Natural Science, 1989.

White, Peter T. "The Fascinating World of Trash." *National Geographic* 163 (April 1983): 424–57.

Wrhen, Linda, and Michael A. DiSpezio. "Rethink, Rework, Recycle." *Science and Children* 28 (April 1991): 33–36.

### Fractals

Banchoff, Thomas F. "Dimension." In *On the Shoulders of Giants: New Approaches to Numeracy*, edited by Lynn A. Steen, pp. 11–59. Washington, D.C.: National Academy Press, 1990.

Camp, Dane R. "A Fractal Execursion." *Mathematics Teacher* 81 (April 1991): 265–75.

Devaney, Robert. *Chaos, Fractals, and Dynamics: Computer Experiments in Mathematics.* Menlo Park, Calif.: Addison-Wesley Publishing Co., 1990.

Gleick, James. *Chaos: The Making of a New Science.* New York: Viking, 1987.

Kern, Jane F., and Cherry C. Mauk. "Exploring Fractals—a Problem-Solving Adventure Using Mathematics and Logo." *Mathematics Teacher* 83 (March 1990): 179–85, 244.

Mandelbrot, Benoit B. *The Fractal Geometry of Nature.* New York: W. H. Freeman & Co., 1982.

Martin, Tami. "Fracturing Ideas about Dimension." *Student Math Notes* [National Council of Teachers of Mathematics], November 1991.

Peitgen, Heinz-Otto, Harmut Jurgens, Dietmar Saupe, Evan Maletsky, Terry Perciante, and Lee Yunker. *Fractals for the Classroom: Strategic Activities*, Vol. 1. New York: Springer-Verlag, 1991.

Peitgen, Heinz-Otto, and Peter H. Richter. *The Beauty of Fractals.* New York: Springer-Verlag, 1986.

Peterson, Ivars. *The Mathematical Tourist: Snapshots of Modern Mathematics.* New York: W. H. Freeman & Co., 1988.

Steen, Lynn A., ed. *On the Shoulders of Giants: New Approaches to Numeracy.* Washington, D.C.: National Academy Press, 1990.

Stillwell, John. *Mathematics and Its History.* New York: Springer-Verlag, 1989.

Curriculum and Evaluation Standards for School Mathematics Addenda Series
Bonnie H. Litwiller, Addenda Series Project Coordinator

**Grades K–6 Addenda Series**
Miriam A. Leiva, Series Editor

Seven grade-level books,
kindergarten through sixth grade
GEOMETRY AND SPATIAL SENSE
MAKING SENSE OF DATA
NUMBER SENSE AND OPERATIONS
PATTERNS

**Grades 5–8 Addenda Series**
Frances R. Curcio, Series Editor

DEALING WITH DATA AND CHANCE
DEVELOPING NUMBER SENSE IN THE MIDDLE GRADES
GEOMETRY IN THE MIDDLE GRADES
MEASUREMENT IN THE MIDDLE GRADES
PATTERNS AND FUNCTIONS
UNDERSTANDING RATIONAL NUMBERS AND PROPORTIONS

**Grades 9–12 Addenda Series**
Christian R. Hirsch, Series Editor

ALGEBRA IN A TECHNOLOGICAL WORLD
CONNECTING MATHEMATICS
A CORE CURRICULUM: MAKING MATHEMATICS COUNT FOR EVERYONE
DATA ANALYSIS AND STATISTICS ACROSS THE CURRICULUM
GEOMETRY FROM MULTIPLE PERSPECTIVES

BEST COPY AVAILABLE